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CANADIAN JOURNAL OF MATHEMATICS

Journal Canadien de Mathématiques

UNIVERSITY OF MICHIGAN
JAN 30 1951
MATHEMATICS LIBRARY

VOL. III - NO. 1

1951

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Published for

THE CANADIAN MATHEMATICAL CONGRESS

by the

University of Toronto Press

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The chief languages of the *Journal* are English and French.

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All other correspondence should be addressed to the *Managing Editor*, G. de B. Robinson, University of Toronto.

The *Journal* is published quarterly. Subscriptions should be sent to the *Managing Editor*. The price per volume of four numbers is $6.00. This is reduced to $3.00 for individuals who are members of the following Societies:

Canadian Mathematical Congress
American Mathematical Society
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The Canadian Mathematical Congress gratefully acknowledges the assistance of the following towards the cost of publishing this Journal:

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THE HAMILTONIAN FORM OF FIELD DYNAMICS

P. A. M. DIRAC

1. Introduction. In classical dynamics one has usually supposed that when one has solved the equations of motion one has done everything worth doing. However, with the further insight into general dynamical theory which has been provided by the discovery of quantum mechanics, one is lead to believe that this is not the case. It seems that there is some further work to be done, namely to group the solutions into families (each family corresponding to one principal function satisfying the Hamilton-Jacobi equation). The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood.

The importance of the family is brought out by the Schrödinger form of quantum mechanics and not by the Heisenberg form. The latter is in direct analogy with the classical Hamiltonian equations of motion and does not require any grouping of the solutions. The Schrödinger form goes beyond this in ascribing importance to the concept of a quantum state, subject to the principle of superposition and described by a solution of Schrödinger's wave equation, and this concept requires the introduction of families of solutions for its analogue in classical mechanics, the Schrödinger equation itself being the analogue of the Hamilton-Jacobi equation.

One can build up a relativistic dynamical theory by starting with a Lorentz invariant action function involving field quantities. The requirement that the total action shall be stationary under arbitrary small variations of those field variables that play the role of dynamical coordinates at all points of space-time leads to a relativistic set of field equations as the equations of motion. These equations may be put in the Hamiltonian form, and one can then pass from them to Heisenberg's form of quantum mechanics. This has already been done by Weiss [1]. The present paper is concerned with the further mathematical development, connected with the grouping of the solutions into families, which is needed before one can pass to Schrödinger's form of quantum mechanics.

The dynamical variables of the Hamiltonian equations of motion must be a set of variables that can serve as initial conditions—they must be independent of one another and sufficient to fix the state of motion. In non-relativistic theory one takes them to be physical quantities referring to an instant of time.

Received August 2, 1950. This paper is based on lectures given at the Canadian Mathematical Seminar in Vancouver in August and September 1949.

The concept of an instant of time is rather artificial from the relativistic point of view. It is to be pictured as a flat three-dimensional "surface" in four-dimensional space-time, with the direction of its normal lying within the light-cone. It would be more natural in a relativistic theory to replace the flat surface by an arbitrary curved one, subject to the restriction that it is everywhere space-like, i.e. the normal at every point of it lies within the light-cone. One would then work with dynamical variables referring to physical conditions on such a curved surface, as was done by Weiss.

The use of a curved surface instead of a flat one of course increases the complexity of the mathematical equations. In working out practical examples one would always revert to the flat surface to simplify the calculations as much as possible, the flat surface being adequate for describing all experimental results. The curved surface is desirable in a general theory because of the flexibility and mathematical power that it gives. It shows up the transformation properties of the Hamiltonian theory applied to field dynamics. In any problem which involves seeking for a new dynamical system, rather than working out the properties of a given dynamical system, it would be advantageous to use the curved surface, because it brings more conditions into the mathematics and so restricts the region of search.

The curved surface will be described by certain mathematical variables, which we shall call the *surface variables*. (Actually, they will consist of functions, as will be discussed in the next section.) The equations of motion will give the change in the dynamical variables when the surface is moved in space-time. The surface can be subjected to arbitrary changes of direction and deformations during the motion, provided it remains always space-like. Thus the equations of motion give the change in the dynamical variables for any change in the surface variables. One can get these equations in the Hamiltonian form by working from the Lagrangian, as was shown by Weiss, and one then has Poisson bracket relations between the dynamical variables.

Let us now consider what development is needed to make possible the grouping of the solutions of the equations of motion into families. One can infer from analogy with non-relativistic dynamics that the Hamilton-Jacobi equation, whose solutions define the families, is a partial differential equation of the first order in the dynamical coordinates, and also in the surface variables. It thus involves the surface variables in the same way as the dynamical coordinates. This provides the key to the problem. *We must put the dynamical theory into a form in which the surface variables are treated on the same footing as dynamical coordinates.* They must have conjugate momenta and there must be P. b. relations connecting them with other dynamical variables. When we have done this, the setting up of the Hamilton-Jacobi equation is straightforward, and the ground is prepared for Schrödinger's form of quantization.

The equations of motion must now be considered in the first place as making all the dynamical variables, including the surface variables, vary together, and giving them all as functions of some independent variable, τ say, when suitable

initial conditions are prescribed. Actually, the surface variables can vary arbitrarily with τ. This is to be taken into account by supposing the general solution of the equations of motion to involve some arbitrary functions (or functionals). The arbitrariness in the motion of the surface variables is then to be ascribed to the "accidental" appearance of these arbitrary functions in the solution of equations which, in a general way, one would normally expect to fix the motion completely.

The generalization of Hamiltonian dynamics which is needed in order that arbitrary functions may appear in the solution of the equations of motion has been given in a previous paper by the author [2]. The present paper is a direct application of the method given there.

2. The general space-like surface. We describe space-time by the four coordinates $x_\mu(\mu = 0, 1, 2, 3)$ of a rectilinear orthogonal system of coordinates. For simplicity we shall write all vectors referred to this coordinate system in the contravariant form, such as a_μ and will make the convention

$$a_\mu b_\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3,$$

applying whenever one of these contravariant suffixes is repeated in a term.

We can describe a general three-dimensional surface in space-time by giving the four coordinates x_μ of any point on it as functions of three parameters $u_r(r = 1, 2, 3)$, say

$$x_\mu = y_\mu(u). \tag{1}$$

This involves setting up a parametrization u on the surface. The parametrization is not necessary physically and brings some extra complication into the mathematics. It could be avoided by using a different method of description of the surface, specifying x_0 as a function of x_1, x_2, x_3. However, this would spoil the relativistic treatment of the four x's. For the sake of relativistic form it seems preferable to use the method (1), the extra complication arising from the parametrization being not very troublesome if properly handled.

The condition that the surface shall be space-like is that there shall exist at every point on it a unit normal vector l_μ, satisfying

$$l_\mu y_\mu{}^r = 1, \qquad l_\mu l_\mu = 1, \qquad l_0 > 0, \tag{2}$$

where $y_\mu{}^r$ is short for the derivative $\partial y_\mu / \partial u_r$. The l_μ are to be understood as functions of the parameters u.

The parameters u form a system of coordinates in the three-dimensional surface. The metric of the surface referred to these coordinates is

$$ds^2 = dy_\mu dy_\mu = y_\mu{}^r du_r y_\mu{}^s du_s = \gamma^{rs} du_r du_s$$

with

$$\gamma^{rs} = \gamma^{sr} = y_\mu{}^r y_\mu{}^s. \tag{3}$$

For a space-like surface this ds^2 is negative and thus the determinant of the γ^{rs} is negative. Put it equal to $-\Gamma^2$, Γ being positive. Then an element of three-dimensional "area" of the surface is

$$\Gamma du_1 du_2 du_3 = \Gamma d^3u$$

say.

Let γ_{rs} be the cofactor of γ^{rs} in the determinant, divided by the determinant, so that

$$\gamma_{rs}\gamma^{sp} = \delta_s^{\ p}.$$

The γ^{rs},γ_{rs} can be used for raising and lowering the suffixes of vector and tensor quantities in the three-dimensional space of the surface. We shall use generally the notation of adding an upper suffix ^{r} to a quantity to denote its derivative with respect to u_r (ordinary, not covariant derivative). Thus with ζ any function of the u's, $\zeta^r = \partial\zeta/\partial u_r$, and then $\zeta_r = \gamma_{rs}\zeta^s$. We have $\zeta^{rs} = \zeta^{sr}$, but in general $\zeta_{rs} \neq \zeta_{sr}$.

We have

$$\partial\Gamma^2/\partial u_p = \Gamma^2\gamma_{rs}\partial\gamma^{rs}/\partial u_p$$

and hence

$$\Gamma^p = \tfrac{1}{2}\Gamma\gamma_{rs}\gamma^{rsp}. \tag{4}$$

Let a_μ be any 4-vector located at a point on the surface. It has a normal component

$$a_\mu l_\mu = a_l$$

say, and a contravariant component in the direction u_r in the surface

$$a_\mu y_\mu^{\ r} = a^r$$

say. The covariant components in the surface directions may be introduced by $a_r = \gamma_{rs}a^s$. It is now easily seen that

$$(a_\mu - a_l l_\mu - a_r y_\mu^{\ r})\ l_\mu = 0$$

$$(a_\mu - a_l l_\mu - a_r y_\mu^{\ r})\ y_\mu^{\ s} = 0,$$

so that the 4-vector $a_\mu - a_l l_\mu - a_r y_\mu^{\ r}$ has its normal component and its components in the directions u_r all vanishing. This means the vector itself vanishes, i.e.

$$a_\mu = a_l l_\mu + a_r y_\mu^{\ r}. \tag{5}$$

This equation expresses the resolution of the vector a_μ into its normal and tangential parts. It is convenient to write the tangential part $a_r y_\mu^{\ r}$ as $a_{-\mu}$ for brevity.

The scalar product of two 4-vectors a_μ and b_μ is

$$\begin{aligned} a_\mu b_\mu &= a_l b_l + a_{-\mu} b_{-\mu} = a_l b_l + a_\tau y_\mu{}^\tau b_\sigma y_\mu{}^\sigma \\ &= a_l b_l + \gamma^{\tau\sigma} a_\tau b_\sigma \\ &= a_l b_l + a_\tau b^\tau. \end{aligned} \tag{6}$$

This result may be written as

$$g_{\mu\nu} a_\mu b_\nu = a_\mu l_\mu b_\nu l_\nu + a_\mu y_{\mu\tau} b_\nu y_\nu{}^\tau.$$

Since this holds for arbitrary a_μ and b_ν, we can infer

$$g_{\mu\nu} = l_\mu l_\nu + y_{\mu\tau} y_\nu{}^\tau, \tag{7}$$

a fundamental formula which is frequently useful.

If $V(u)$ is any function defined on the surface, we can obtain by differentiation the 3-vector $\partial V/\partial u_\tau = V^\tau$. We may then form $V^\tau y_{\mu\tau} = V_{-\mu}$ and consider it as a 4-vector in space-time. If V is a field function, so that it is defined off the surface as well as on the surface, we can form $\partial V/\partial x_\mu$ and, by changing the sign of its spatial components, obtain the contravariant 4-vector V_μ. Then the above $V_{-\mu}$ is just the tangential part of V_μ.

Differentiating equations (2) with respect to u_σ, we get

$$l_\mu{}^\sigma y_\mu{}^\tau + l_\mu y_\mu{}^{\tau\sigma} = 0, \tag{8}$$

$$l_\mu l_\mu{}^\sigma = 0. \tag{9}$$

Equation (8) shows that

$$l_\mu{}^\sigma y_\mu{}^\tau = l_\mu{}^\tau y_\mu{}^\sigma.$$

Define

$$\Omega^{\tau\sigma} = \Omega^{\sigma\tau} = l_\mu{}^\tau y_\mu{}^\sigma. \tag{10}$$

It is the curvature tensor. It may be expressed in the four dimensional form

$$\begin{aligned} \Omega_{-\mu-\nu} &= y_{\mu\tau} y_{\nu\sigma} \Omega^{\tau\sigma} = y_{\mu\tau} y_{\nu\sigma} l_\sigma{}^\tau y_\sigma{}^\sigma \\ &= y_{\mu\tau} l_\sigma{}^\tau (g_{\sigma\nu} - l_\sigma l_\nu). \end{aligned}$$

with the help of (7). Using (9) we now get

$$\Omega_{-\mu-\nu} = y_{\mu\tau} l_\nu{}^\tau = l_{\nu-\mu}. \tag{11}$$

We can infer that

$$l_{\mu-\nu} = l_{\nu-\mu}. \tag{12}$$

3. Poisson bracket relations. The $y_\mu(u)$ of (1), considered as a set of numbers with μ and the u's taking all permissible values, are the surface variables. They are to be treated as dynamical coordinates. Since the u's take on a continuous range of values, these variables will give us a continuous range of degrees of freedom, instead of the usual discrete set. The formalism of paper [2] was set up for a dynamical system with a discrete set of degrees of freedom,

but can be made to apply to a continuous range by replacing sums by integrals and the Kronecker δ-symbol by a δ-function. The δ-function we now need is $\delta(u - u')$, which vanishes for $u \neq u'$ and has its integral

$$\int \delta(u - u')d^2u = 1. \tag{13}$$

Note that there is no factor Γ occurring with the d^2u, so this δ-function refers to the parametrization of the surface and not to the metric. If one did introduce the factor Γ into (13) one would get a δ-function with a more directly physical meaning, but it would not be so suitable for dynamical theory because it would not have zero P.b. with the w variables introduced below.

The dynamical variables $y_\mu(u)$ have conjugate momenta, $w_\mu(u)$ say, satisfying the standard P.b. relations. If for brevity we write y_μ, y'_μ for $y_\mu(u)$, $y_\mu(u')$ and similarly w_μ, w'_μ for $w_\mu(u)$, $w_\mu(u')$, these relations are

$$[y_\mu, y'_\nu] = 0, \quad [w_\mu, w'_\nu] = 0, \quad [y_\mu, w'_\nu] = g_{\mu\nu}\delta(u - u'). \tag{14}$$

The momenta may be pictured as associated with displacements and deformations of the parametrized surface, the linear combination $\epsilon\int a_\mu w_\mu d^2u$, where a_μ is a function of u and ϵ is small, being associated with the displacement in which y_μ is changed to $y_\mu + \epsilon a_\mu$ for all u values. Thus the normal component $w_\mu l_\mu = w_l$ is associated with a motion of the surface normal to itself, and the tangential components $w_\mu y_\mu{}^r = w^r$ with merely changes in the parametrization.

From the fundamental P.b. relations (14) one can deduce a number of useful P.b. relations connecting the w variables with functions of the surface variables and with each other. Some of these relations have been obtained previously by the author [3], working from the association of the w variables with deformations, and by Chang [4], using a more direct method similar to the one used below. Note that the Π^r,Π^s of these two papers are minus the present w^r, w_l and the γ^{rs}, γ_{rs} of [3] are minus the present γ^{rs}, γ_{rs}.

In working out P.b.'s one should note in the first place that all quantities depending only on the surface and its parametrization are functions of the dynamical coordinates $y_\mu(u)$ only, and so have zero P.b.'s with one another. Thus y_μ, l_μ, $y_\mu{}^r$, Ω^{rs} for all u values, and all their derivatives with respect to u's, have zero P.b.'s with one another.

We have from the first of equations (2)

$$0 = [l_\mu y_\mu{}^r, w'_\nu] = [l_\mu, w'_\nu]y_\mu{}^r + l_\mu[y_\mu{}^r, w'_\nu].$$

Now
$$[y_\mu{}^r, w'_\nu] = [y_\mu, w'_\nu]^r$$

$$= g_{\mu\nu}\delta^r(u - u'). \tag{15}$$

Hence
$$[l_\mu, w'_\nu]y_\mu{}^r = -l_\nu\delta^r(u - u').$$

Again, from the second of equations (2),

$$0 = \tfrac{1}{2}[l_\mu l_\mu, w'_\nu] = [l_\mu, w'_\nu]\, l_\mu.$$

Thus, using (7),

$$[l_\lambda, w'_\nu] = [l_\mu, w'_\nu](l_\mu l_\lambda + y_\mu{}^\rho y_{\lambda\rho})$$

$$= - l_\nu y_{\lambda\rho}\delta^\rho(u - u') = - l_\nu \delta_{-\lambda}(u - u'). \tag{16}$$

We have, using (15),

$$[\gamma^{\rho\sigma}, w'_\nu] = y_\mu{}^\rho[y_\mu{}^\sigma, w'_\nu] + y_\mu{}^\sigma[y_\mu{}^\rho, w'_\nu] = y_\nu{}^\rho\delta^\sigma(u - u') + y_\nu{}^\sigma\delta^\rho(u - u'). \tag{17}$$

Following the method by which (4) was obtained, we get

$$[\Gamma^2, w'_\nu] = \Gamma^2\gamma_{\rho\sigma}[\gamma^{\rho\sigma}, w'_\nu] = 2\Gamma^2\gamma_{\rho\sigma}y_\nu{}^\rho\delta^\sigma(u - u')$$

$$= 2\Gamma^2 y_{\nu\sigma}\delta^\sigma(u - u') = 2\Gamma^2\delta_{-\nu}(u - u'),$$

or

$$[\Gamma, w'_\nu] = \Gamma\delta_{-\nu}(u - u'). \tag{18}$$

From (17) again

$$[\gamma_{\rho\tau}, w'_\nu]\gamma^{\tau\sigma} = - \gamma_{\rho\tau}[\gamma^{\tau\sigma}, w'_\nu] = - y_{\nu\rho}\delta^\sigma(u - u') - y_\nu{}^\sigma\delta_\rho(u - u'),$$

so

$$[\gamma_{\rho\tau}, w'_\nu] = - y_{\nu\rho}\delta_\tau(u - u') - y_{\nu\tau}\delta_\rho(u - u'). \tag{19}$$

We shall use the notation of adding a dashed suffix τ' or $-\mu'$ to any function X of u' with the following meanings:

$$X^{\tau'} = \partial X/\partial u'_\tau, \qquad X_{\tau'} = \gamma_{\tau\sigma}(u')X^{\sigma'}, \qquad X_{-\mu'} = y'_{\mu\tau'}X^{\tau'}. \tag{20}$$

We get from (16)

$$[l_\lambda, w'_l] = - l'_\nu l_\nu \delta_{-\lambda}(u - u')$$

$$= - \{l'_\nu l_\nu \delta(u - u')\}_{-\lambda} + l'_\nu l_{\nu-\lambda}\delta(u - u')$$

$$= - \{l_\nu l_\nu \delta(u - u')\}_{-\lambda} + l_\nu l_{\nu-\lambda}\delta(u - u')$$

$$= - \delta_{-\lambda}(u - u'). \tag{21}$$

Again, from (16)

$$[l_\lambda, w'^\tau] = - y'_\nu{}^{\tau'} l_\nu \delta_{-\lambda}(u - u')$$

$$= - \{y'_\nu{}^{\tau'} l_\nu \delta(u - u')\}_{-\lambda} + y'_\nu{}^{\tau'} l_{\nu-\lambda}\delta(u - u')$$

$$= y_\nu{}^\tau l_{\nu-\lambda}\delta(u - u')$$

$$= y_\nu{}^\tau l_{\lambda-\nu}\delta(u - u')$$

$$= l_\lambda{}^\tau\delta(u - u'). \tag{22}$$

Similarly, we get from (18)

$$[\Gamma, w'_l] = l'_\nu\Gamma\delta_{-\nu}(u - u') = \Gamma\{l'_\nu\delta(u - u')\}_{-\nu}$$

$$= \Gamma l_{\nu-\nu}\delta(u - u'), \tag{23}$$

and again from (18)

$$[\Gamma,w'^r] = y'_{\nu}{}^{r'}\Gamma\delta_{-\nu}(u-u') = \Gamma\{y'_{\nu}{}^{r'}\delta(u-u')\}_{-},$$
$$= \Gamma(y_{\nu}{}^{r})_{-\nu}\delta(u-u') + \Gamma y_{\nu}{}^{r}\delta_{-\nu}(u-u').$$

Now $$(y_{\nu}{}^{r})_{-\nu} = y_{\nu}{}^{rs}y_{\nu}{}^{p}\gamma_{ps} = y_{\nu}{}^{sr}y_{\nu}{}^{p}\gamma_{ps}$$
$$= \tfrac{1}{2}(y_{\nu}{}^{s}y_{\nu}{}^{p})^{r}\gamma_{ps} = \tfrac{1}{2}\gamma^{ps r}\gamma_{ps}$$

(24) $$= \Gamma^r/\Gamma$$

from (4). Hence

$$[\Gamma,w'^r] = \Gamma^r\delta(u-u') + \Gamma\delta^r(u-u')$$

(25) $$= \{\Gamma\delta(u-u')\}^r.$$

To get the P.b.'s of w_l, w^r, we have

$$[w_l,w'_l] = [w_\mu l_\mu, w'_\nu l'_\nu]$$
$$= w_\mu l'_\nu[l_\mu,w'_\nu] + l_\mu w'_\nu[w_\mu,l'_\nu]$$
$$= w_\mu[l_\mu,w'_l] + w'_\nu[w_l,l'_\nu].$$

From (21) this equals

(26) $$[w_l,w'_l] = -w_\mu\delta_{-\mu}(u-u') + w'_\nu\delta_{-\nu}(u-u').$$

Again $$[w_l,w'^r] = [w_\mu l_\mu, w'_\nu y'_\nu{}^{r'}]$$
$$= w_\mu[l_\mu,w'^r] + l_\mu w'_\nu[w_\mu,y'_\nu{}^{r'}]$$
$$= w_\mu l_\mu{}^r\delta(u-u') + l_\mu w_\mu{}'\delta^r(u-u')$$

with the help of (22). This gives

$$[w_l,w'^r] = w_\mu l_\mu{}^r\delta(u-u') + l_\mu\{w'_\mu\delta(u-u')\}^r$$

(27) $$= \{w_l\delta(u-u')\}^r.$$

Again $$[w^r,w'^s] = [w_\mu y_\mu{}^r, w'_\nu y'_\nu{}^{s'}]$$
$$= w_\mu y'_\nu{}^{s'}[y_\mu{}^r,w'_\nu] + y_\mu{}^r w'_\nu[w_\mu,y'_\nu{}^{s'}]$$
$$= w_\mu y'_\mu{}^{s'}\delta^r(u-u') - w'_\mu y_\mu{}^r\delta^{s'}(u-u')$$
$$= w_\mu\{y_\mu{}^{sr}\delta(u-u') + y_\mu{}^s\delta^r(u-u')\} - w_\mu{}'\{y'_\mu{}^{r's'}\delta(u-u') + y'_\mu{}^{r'}\delta^{s'}(u-u')\}$$

(28) $$= w^s\delta^r(u-u') - w'^r\delta^{s'}(u-u').$$

Further useful relations are

$$[y_{\mu s},w'_\nu] = [\gamma_{rs}y_\mu{}^r,w'_\nu]$$
$$= \gamma_{rs}g_{\mu\nu}\delta^r(u-u') - y_\mu{}^r\{y_{\nu r}\delta_s(u-u') + y_{\nu s}\delta_r(u-u')\}$$

(29) $$= l_\mu l_\nu\delta_s(u-u') - y_{\nu s}\delta_{-\mu}(u-u')$$

with the help of (7), and

$$
\begin{aligned}
[y_{\mu s}, w'^{r}] &= y'_{s}{}^{r}\{l_{\mu}l_{\nu}\delta_{s}(u-u') - y_{\nu s}y_{\mu p}\delta^{p}(u-u')\} \\
&= y'_{s}{}^{r}l_{\mu}\{l'_{\nu}\delta_{s}(u-u') - l_{\nu s}\delta(u-u')\} - y_{\mu p}y_{\nu s}\{y_{\nu}{}^{p}\delta(u-u') \\
&\qquad + y_{\nu}{}^{r}\delta^{p}(u-u')\} \\
&= -l_{\mu}l_{\nu}{}^{r}y_{\nu s}\delta(u-u') - y_{\mu p}y_{\nu}{}^{pr}y_{\nu s}\delta(u-u') - \delta^{r}{}_{s}\delta_{-\mu}(u-u') \\
&= l_{\mu}l_{\nu}y_{\nu s}{}^{r}\delta(u-u') + y_{\mu p}y_{\nu}{}^{p}y_{\nu s}{}^{r}\delta(u-u') - \delta^{r}{}_{s}\delta_{-\mu}(u-u') \\
&= y_{\mu s}{}^{r}\delta(u-u') - \delta_{s}{}^{r}\delta_{-\mu}(u-u') \qquad (30)
\end{aligned}
$$

with the help of (7) again.

4. Changes of parametrization. Using the homogeneous velocity formulation of the dynamical equations, we have a Hamiltonian of the general form given by the equation (20) of paper [2]. The ϕ's here are functions of the dynamical coordinates and momenta and the v's involve the τ derivatives of quantities that can vary arbitrarily with τ. With our present dynamical system we have the surface variables $y_{\mu}(u)$ that can vary arbitrarily with τ and may take their τ derivatives to be v's. (Alternatively, we could take any complete set of independent functions of the $\dot{y}_{\mu}$'s to be v's and get an equivalent but less convenient formulation). If these are the only quantities that can vary arbitrarily with τ, their τ derivatives are the only v's and the Hamiltonian is

$$H \equiv \int \dot{y}_{\mu}\phi_{\mu}(u)d^{3}u, \qquad (31)$$

with $\phi_{\mu}(u)$ some function of the dynamical coordinates and momenta, weakly equal to zero, for each value of u If there are other quantities that can vary arbitrarily with τ, there will be further terms in H. These will be left understood for the present, as they will not affect the arguments now being used.

From (6) we may write (31) alternatively in terms of the normal and tangential components of $\dot{y}_{\mu}$ and ϕ_{μ},

$$H \equiv \int \dot{y}_{l}\phi_{l}d^{3}u + \int \dot{y}_{r}\phi^{r}d^{3}u. \qquad (32)$$

We now have ϕ_{l} and ϕ^{r} functions of the dynamical coordinates and momenta weakly equal to zero.

According to equation (21) of paper [2], the equation of motion for a general dynamical variable g is

$$\dot{g} = \int \dot{y}_{\mu}[g,\phi_{\mu}]d^{3}u, \qquad (33)$$

or alternatively

$$\dot{g} = \int \dot{y}_{l}[g,\phi_{l}]d^{3}u + \int \dot{y}_{r}[g,\phi^{r}]d^{3}u. \qquad (34)$$

The second term here gives the change in g when $\dot{y}_{l} = 0$, which means that the surface itself does not move but only its parametrization changes. Thus for a small change in parametrization

$$dg = \int (dy)_r [g, \phi^r] d^3u. \tag{35}$$

A small change in the parametrization is given by

$$u_r \to u_r + \epsilon a_r, \tag{36}$$

meaning that the point on the surface with parameters u_r becomes the point with parameters $u_r + \epsilon a_r$, where a_r is a function of the u's and ϵ is a small number. This makes $y_\mu(u)$ change by

$$dy_\mu = y_\mu(u + \epsilon a) - y_\mu(u) = \epsilon y_\mu{}^r a_r$$

and gives

$$(dy)_r = y_{\mu r} dy_\mu = \epsilon a_r.$$

The change in g for this change in the parametrization is thus

$$dg = \epsilon \int a_r [g, \phi^r] d^3u \tag{37}$$

$$= \epsilon [g, \int a_r \phi^r d^3u], \tag{37'}$$

since $\phi^r = 0$. The result (37) or (37′) holds even if a_r is a function of any of the dynamical variables. We see now the importance of the ϕ^r's as the quantities with which one must form the P.b. of any dynamical variable to get its change under a change in the parametrization.

If a quantity refers to a point u' on the surface and is invariant under any change of the parametrization that leaves the point u' invariant, I call it a *u-scalar at the point u'*. A quantity that is invariant under any change of the parametrization whatever I call a *u-invariant*. The concepts of u-scalar and u-invariant refer only to the dependence on transformations of the u's and not on how the quantity behaves under a Lorentz transformation. A u-scalar or u-invariant may very well be a component of a vector or tensor so far as Lorentz transformations are concerned.

Now $y_\mu(u')$ for a particular value of μ is evidently a u-scalar at u'. So is $l_\mu(u')$. Any function of u-scalars at u' is a u-scalar at u'. If $S(u')$ is any u-scalar at u', then $\int S(u') \Gamma' d^3u'$ is a u-invariant.

Let $S(u)$ be a u-scalar at the point u. Under the change of parametrization (36) it will change to the same u-scalar at $u + \epsilon a$, namely $S(u + \epsilon a)$, so that

$$dS = S(u + \epsilon a) - S(u) = \epsilon a_r S^r.$$

Thus from (37)

$$a_r S^r = \int a'_r [S, \phi'^r] d^3u'.$$

Since the functions $a_r(u)$ are arbitrary, we can infer

$$[S, \phi'^r] = S^r \delta(u - u'). \tag{38}$$

This is an equation expressing the condition for S to be a u-scalar at u.

Let Q be a u-invariant. From (37) we see that we must have

$$[Q, \phi'^r] = 0 \tag{39}$$

for all u'. Putting $Q = \int S\Gamma d^3u$, we get

$$\int S[\Gamma,\phi'^r]d^3u = -\int\Gamma[S,\phi'^r]d^3u = -\int\Gamma S^r\delta(u-u')d^3u$$
$$= \int S\{\Gamma\delta(u-u')\}^r d^3u.$$

Since this must hold for any u-scalar S, we see that

$$[\Gamma,\phi'^r] = \{\Gamma\delta(u-u')\}^r. \tag{40}$$

Suppose we have a field quantity $V(x)$ having a definite value at every point x of space-time after the equations of motion have been integrated. The values of $V(x)$ on the surface will provide an ∞^3 of numbers, which may be labelled by the parameters u so as to give the function $V(u)$. The quantities $V(u)$ for all values of the u's will be dynamical variables having zero P.b. with all the y and w variables, thus

$$[y_\mu,V'] = 0, \qquad [w_\mu,V'] = 0. \tag{41}$$

They may be dynamical coordinates, in which case they would have zero P.b. with one another,

$$[V,\,V'] = 0. \tag{42}$$

There will then be dynamical variables $U(u)$ say, forming the conjugate momenta to the $V(u)$'s, satisfying the P.b. relations

$$\begin{aligned} &[y_\mu,U'] = 0 \qquad [w_\mu,U'] = 0 \\ &[U,U'] = 0 \qquad [V,U'] = \delta(u-u'). \end{aligned} \tag{43}$$

From its physical meaning $V(u)$ must be independent of changes of parametrization that do not change the point u, so it must be a u-scalar at u. The conjugate momentum $U(u)$ is not also a u-scalar, on account of the dependence of $\delta(u-u')$ in (43) on the parametrization. In order that V may satisfy the condition (38) for a u-scalar, we must have

$$\phi^r \equiv UV^r + \phi^{r+}, \tag{44}$$

where ϕ^{r+} has zero P.b. with all the V's. We may assume that ϕ^{r+} also has zero P.b. with all the U's, this being the simplest assumption leading to a self-consistent scheme, and then

$$\begin{aligned} [U,\phi'^r] &= U'[U,V'^r] = U'\delta^r(u-u') \\ &= \{U\delta(u-u')\}^r. \end{aligned} \tag{45}$$

From (40) we now get

$$\begin{aligned} [U\Gamma^{-1},\phi'^r] &= \{U\delta(u-u')\}^r\Gamma^{-1} - U\Gamma^{-2}\{\Gamma\delta(u-u')\}^r \\ &= \{U\Gamma^{-1}\}^r\delta(u-u'), \end{aligned}$$

which shows, according to (38), that $U\Gamma^{-1}$ is a u-scalar at u. We may call U

itself a u-scalar density. The result that a field quantity that is u-scalar has for its dynamical conjugate a u-scalar density has been obtained previously by Chang [4].

The dynamical coordinates $y_\mu(u)$, being u-scalars, may be treated in the same way as the $V(u)$ above. Corresponding to (44) we can infer that

$$\phi^r \equiv w_\mu y_\mu{}^r + \phi^{r*} \equiv w^r + \phi^{r*}, \tag{46}$$

where ϕ^{r*} has zero P.b. with all the y's. We may assume ϕ^{r*} also has zero P.b. with all the w's. The consistency of this assumption is easily checked. Thus with ϕ^r given by (46), one sees that the P.b. relation (22) leads to the condition, equation (38) with l_λ for S, for l_λ to be a u-scalar; the P.b. relation (25) leads to (40); and

$$[w_\nu,\phi'^r] = w'_\mu g_{\mu\nu}\delta^r(u-u') = \{w_\nu\delta(u-u')\}^r,$$

which checks that w_ν is a u-scalar density.

For a dynamical system in which the only dynamical coordinates are the y's and a number of field quantities $V_a(u)$, $(a = 1, 2, \dots)$, there will be the momentum variables w and the conjugates $U_a(u)$ to the $V_a(u)$, and ϕ^r will be

$$\phi^r \equiv w^r + \textstyle\sum_a U_a V_a{}^r, \tag{47}$$

neglecting an unimportant term which has zero P.b. with all the dynamical variables.

If $S(u)$ is a u-scalar at u, we should expect $S_{-\mu} = y_{\mu r}S^r$ to be also a u-scalar at u, as its formation from $S(u)$ does not depend on the parametrization. A formal proof of this result is as follows. We start from the condition (38) for $S(u)$. Differentiating both sides of this equation with respect to u_α, we get

$$[S^s,\phi'^r] = S^{r\alpha}\delta(u-u') + S^r\delta^\alpha(u-u').$$

Thus
$$\begin{aligned} y_{\mu\alpha}[S^s,\phi'^r] &= \left\{(S^s y_{\mu s})^r - S^s y_{\mu s}{}^r\right\}\delta(u-u') + S^r y_{\mu s}\delta^s(u-u')\} \\ &= \left\{(S_{-\mu})^r - S^s y_{\mu s}{}^r\right\}\delta(u-u') + S^r\delta_{-\mu}(u-u') \\ &= (S_{-\mu})^r\delta(u-u') - S^s[y_{\mu s},\phi'^r] \end{aligned}$$

from (30). Hence

$$[S_{-\mu},\phi'^r] = y_{\mu s}[S^s,\phi'^r] + S^s[y_{\mu s},\phi'^r] = (S_{-\mu})^r\delta(u-u'),$$

which is the condition for $S_{-\mu}$ to be a u-scalar.

5. Passage from the Lagrangian to the Hamiltonian. Consider a dynamical system for which the action density $\mathfrak{L}$ in space-time is a function of certain field quantities V and their first derivatives $\partial V/\partial x^\mu = V_\mu$,

$$\mathfrak{L} \equiv \mathfrak{L}(V, V_\mu).$$

Only one V will be written in the equations of this section for brevity. The action is then

$$I = \int \mathfrak{L} d^4x.$$

An element of four-dimensional "volume" of space-time d^4x can be expressed as an element of three-dimensional "area" of the surface, Γd^3u, multiplied by an element of distance normal to the surface, $l_\mu \dot{y}_\mu d\tau = \dot{y}_l d\tau$. Thus

$$I = \int\int \mathfrak{L} \dot{y}_l \Gamma d^3u d\tau,$$

and so the Lagrangian is

$$L = dI/d\tau = \int \mathfrak{L} \dot{y}_l \Gamma d^3u. \tag{48}$$

This Lagrangian will now be treated according to the general method of paper [2] and the Hamiltonian obtained from it. The equations of sections 2 and 3 can all be used as strong equations in this work.

We must first express L in terms of the dynamical coordinates and velocities, the q's and $\dot{q}$'s of paper [2]. The variables $y_\mu(u)$, $V(u)$ are the q's. Tangential derivatives of y_μ and V, such as $y_\mu{}^\tau$, V^τ, $V_{-\mu}$, are functions of the q's. Derivatives which are not tangential, such as V_μ, are not functions of the q's only, but can be expressed as functions of the q's and $\dot{q}$'s. We have

$$\dot{V} = V_\tau \dot{y}_\tau = (V_l l_\tau + V_{-\tau})\dot{y}_\tau$$

from (5). Thus

$$V_l = (\dot{V} - V_{-\tau}\dot{y}_\tau)/\dot{y}_l, \tag{49}$$

so that

$$V_\mu = V_{-\mu} + l_\mu V_l$$

$$= V_{-\mu} + l_\mu(\dot{V} - V_{-\tau}\dot{y}_\tau)/\dot{y}_l. \tag{50}$$

Here we have V_μ expressed in terms of $\dot{V}$ and $\dot{y}_\mu$, which are $\dot{q}$'s, and $V_{-\mu}, l_\mu$, which are functions of the q's. L now becomes a function of the q's and $\dot{q}$'s. Note that V_μ is homogeneous of degree zero in the velocities, so that L is homogeneous of the first degree in the velocities, as is needed for the homogeneous velocity formulation of the dynamical equations.

If we vary the $\dot{q}$'s keeping the q's constant, we get from (50)

$$\delta V_\mu = l_\mu(\delta\dot{V} - V_{-\tau}\delta\dot{y}_\tau)/\dot{y}_l - l_\mu(\dot{V} - V_{-\tau}\dot{y}_\tau)\delta\dot{y}_l/\dot{y}_l{}^2$$

$$= l_\mu(\delta\dot{V} - V_\tau\delta\dot{y}_\tau)/\dot{y}_l$$

using (50) again. The variation in L given by (48) is then

$$\delta L = \int\{\partial\mathfrak{L}/\partial V_\mu \,.\, \delta V_\mu \dot{y}_l + \mathfrak{L}\delta\dot{y}_l\}\Gamma d^3u$$

$$= \int\{\partial\mathfrak{L}/\partial V_\mu \,.\, l_\mu(\delta\dot{V} - V_\tau\delta\dot{y}_\tau) + \mathfrak{L} l_\tau \delta\dot{y}_\tau\}\Gamma d^3u. \tag{51}$$

From the definition of the momenta $w_\mu(u)$, $U(u)$ conjugate to $y_\mu(u)$, $V(u)$ respectively, equation (2) of paper [2] applied to a continuous range of degrees of freedom, we have

$$\delta L = \int \{w_r \delta \dot{y}_r + U \delta \dot{V}\} d^3u. \tag{52}$$

Comparing (51) and (52), we find

$$U = \partial \mathfrak{L} / \partial V_\mu \,.\, l_\mu \Gamma, \tag{53}$$

$$w_r = - \partial \mathfrak{L} / \partial V_\mu \,.\, l_\mu V_r \Gamma + \mathfrak{L} l_r \Gamma = - UV_r + \mathfrak{L} l_r \Gamma. \tag{54}$$

These are the weak equations giving the momenta in terms of the q's and $\dot{q}$'s.

Following the method of paper [2], we eliminate the $\dot{q}$'s from equations (53) and (54) so as to get weak equations involving only the p's (i.e. the w's and U's) and the q's. These are the ϕ equations. Equation (54) is best treated by splitting it into a normal part, obtained by multiplying it by l_r,

$$w_l + UV_l - \mathfrak{L} \Gamma = 0, \tag{55}$$

and a tangential part, obtained by multiplying it by y_r^r,

$$w^r + UV^r = 0. \tag{56}$$

There are no $\dot{q}$'s in (56), so (56) as it stands is a ϕ equation. Its left-hand side is just the ϕ^r of (47) associated with changes of parametrization, the summation sign being understood in (56).

Equations (53) and (55) involve the derivatives V_μ of V, which derivatives can be expressed in terms of $V_{-\mu}$ and V_l. The $\dot{q}$'s then occur in (53) and (55) only through V_l, that is, from (49), only through the combination $(\dot{V} - V_{-r}\dot{y}_r)/\dot{y}_l$.

For many dynamical systems (see example 1 below) one can solve equations (53)—there is one of these equations for each field quantity V—to get all the V_l's expressed as functions of the U's and q's. This case will be referred to as the *standard case* in field dynamics; it corresponds to the case in ordinary dynamics with homogeneous velocities when the ratios of the velocities can all be expressed as function of the p's and q's.

In the standard case one can get no ϕ equations from equations (53) alone, but in other cases one can get ϕ equations from (53) alone. In the standard case one can get a ϕ equation from (55) with the help of (53), namely the equation

$$w_l + \mathfrak{H} = 0, \tag{57}$$

where $\mathfrak{H}$ is the result of substituting for V_l in $UV_l - \mathfrak{L}\Gamma$ its value in terms of the U's and q's given by (53). Equations (57) and (56), taken for all u-values, are then the only ϕ equations. In other cases one can still get a ϕ equation like (57) from (55) with the help of (53), as will be shown later. This equation, together with (56) and the ϕ equations which follow from (53) alone, are then the only ϕ equations.

The field equations are obtained in the usual way from the variation of the action integral and are

$$\frac{\partial}{\partial x_\mu} \frac{\partial \mathfrak{L}}{\partial V_\mu} = \frac{\partial \mathfrak{L}}{\partial V}. \tag{58}$$

These equations must be examined to see whether they lead to any equations between p's and q's only. Such equations would be χ equations. With the help of (7) and (53), (58) may be written as

$$\frac{l_\mu}{\Gamma}\left\{\frac{\partial U}{\partial x_\mu} - \frac{\partial \mathfrak{L}}{\partial V_r}\frac{\partial(l_r\Gamma)}{\partial x_\mu}\right\} + y_{\mu r}\frac{\partial}{\partial u_r}\frac{\partial \mathfrak{L}}{\partial x_\mu} = \frac{\partial \mathfrak{L}}{\partial V}, \tag{59}$$

and thus involves the normal derivative of U. In the standard case the U's are all independent functions of the velocities and one cannot eliminate their normal derivatives from (59), so there can be no χ equations. In other cases, however, there may well be χ equations (see examples 2 and 3 below).

When we have obtained all the ϕ's and χ's we must see which of them are first class, that is, have zero P.b. with all the ϕ's and χ's. This can always be decided by working out the P.b's using the results of section 3, but we can infer that some ϕ's are first class more easily by observing that they occur in the Hamiltonian, giving rise to arbitrary functions in the general solution of the equations of motion.

The Hamiltonian is, from the definition (5) of paper [2],

$$H = \int (w_\mu \dot{y}_\mu + U\dot{V} - \mathfrak{L}\dot{y}_l\Gamma)d^2u. \tag{60}$$

It may be written

$$\begin{aligned} H &= \int \{w^r\dot{y}_r + w_l\dot{y}_l + U(V_l l_r + V_{-r})\dot{y}_r - \mathfrak{L}\dot{y}_l\Gamma\}d^2u \\ &= \int \{\dot{y}_l(w_l + UV_l - \mathfrak{L}\Gamma) + \dot{y}_r(w^r + UV^r)\}d^2u \\ &= \int \dot{y}_l(w_l + UV_l - \mathfrak{L}\Gamma)d^2u + \int \dot{y}_r\phi^r d^2u, \end{aligned} \tag{61}$$

with ϕ^r given by (47), the summation sign being now understood. According to the general theory of paper [2], H must strongly equal a linear function of first class ϕ's of the form (31) or (32), with extra terms if there are other quantities besides the surface variables $y_\mu(u)$ that can vary arbitrarily with τ. The ϕ^r of (61) is the same as the ϕ^r of (32), so we can infer that it must be first class. The presence of this first class ϕ in H gives rise to arbitrary changes of parametrization during the motion.

In the standard case the only ϕ equation, apart from the ϕ^r equation (56), is equation (57). Hence (57) must be the same equation as $\phi_l = 0$. We can infer that the left-hand side of (57) must be first class, and also that the equation

$$\mathfrak{H} \equiv UV_l - \mathfrak{L}\Gamma, \tag{62}$$

which is needed to make the first term of (61) go over into the first term of the right-hand side of (32), holds strongly, and not merely weakly, as we knew previously. Further, we can infer that in the standard case there are no extra terms in H besides the ones appearing in (32), so there are no other quantities that can vary arbitrarily with τ besides the surface variables.

In other cases there must still be a ϕ equation (57), deducible from (55) with the help of (53), in order that its left-hand side may be the first class ϕ_l, whose presence in H gives rise to the arbitrary motion of the surface normal to itself. Equation (62) still holds strongly if there are no extra terms in H (see example 2 below). However if there are extra terms in H, as is the case when there are first class ϕ equations deducible from (53) alone, the first term of (61) no longer strongly equals the first term of the right-hand side of (32) and so (62) no longer holds strongly. $\mathfrak{H}$ is then not a uniquely defined quantity, as one can add to it any linear function of the first class ϕ's that follow from (53) alone. (See example 3 below).

To prepare the theory for quantization we must divide all the ϕ and χ equations into first and second class and then change the second class ones into strong equations by a redefinition of P.b.'s, in the way discussed in section 8 of paper [2]. Further, we must change the first class χ's into first class ϕ's, adding them, with arbitrary coefficients, to the Hamiltonian. This change merely involves an increase in the number of solutions of the equations of motion and does not invalidate the existing solutions. (See example 3 below for a discussion of the physical significance of such a change).

We are left with a set of weak first class ϕ equations, from which we can get the Hamilton-Jacobi equations by putting each momentum variable p equal to $\partial S/\partial q$, so that they become first order partial differential equations in S. Their mutual consistency follows from the first class condition. Each of their solutions gives a family of solutions of the equations of motion.

The passage to the quantum theory can now be made according to the rules of section 11 of paper [2]. Each of the weak first class ϕ equations provides one Schrödinger wave equation.

Example 1. The scalar meson field. Some simple examples will now be treated according to the method of the present paper to illustrate various features of the theory. Let us take first the scalar meson field. For this example there is one field quantitiy V, a Lorentz scalar, and the action density is

$$\mathfrak{L} = \tfrac{1}{2} V_\mu V_\mu - \tfrac{1}{2} m^2 V^2, \tag{63}$$

m being a constant.

Equation (53) gives for the momentum U

$$U = V_\mu I_\mu \Gamma = V_l \Gamma. \tag{64}$$

This can be solved to give V_l in terms of U and the q's, so we have the standard case. Thus there can be no χ equations, and the only ϕ besides ϕ^r will be (57). To get the $\mathfrak{H}$ here, we note that (62) and (63) give

$$\mathfrak{H} = UV_l - \tfrac{1}{2}(V_l V_l + V_{-\mu} V_{-\mu} - m^2 V^2)\Gamma.$$

From the weak equation (64) we can infer the strong equation

$$0 = \tfrac{1}{2}(U - V_l \Gamma)^2 \Gamma^{-1}.$$

Adding this to the preceding equation, we obtain

$$\mathfrak{H} \equiv \tfrac{1}{2}U^2\Gamma^{-1} - \tfrac{1}{2}(V_{-\mu}V_{-\mu} - m^2V^2)\Gamma. \tag{65}$$

The normal derivative V_l has disappeared from this expression for $\mathfrak{H}$, so this expression is the correct one, involving only U's and q's, to occur in the ϕ equation (57).

The above derivation of $\mathfrak{H}$ shows that one can eliminate V_l from the right-hand side of (62) using only strong equations. This checks that the equation (62) for $\mathfrak{H}$ is a strong equation, as is necessary in the standard case.

Example 2. The vector meson field. Let us suppose there are four field quantities V forming a Lorentz vector A_μ and let us take the action density

$$\mathfrak{L} \equiv -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}m^2A_\mu A_\mu, \tag{66}$$

where $$F_{\mu\nu} \equiv A_{\nu\mu} - A_{\mu\nu}, \qquad A_{\nu\mu} \equiv \partial A_\nu/\partial x^\mu.$$

There are considerable differences in the treatment of this problem according to whether the constant m is zero or not. In the present example we restrict it to be not zero.

Let B_μ be the momentum conjugate to A_μ, its sign being defined so that

$$[A_\mu,B'_\nu] \equiv g_{\mu\nu}\delta(u - u'). \tag{67}$$

Then (53) gives

$$B_\nu = F_{\nu\mu}l_\mu\Gamma. \tag{68}$$

We can deduce

$$B_l \equiv l_\nu B_\nu = 0. \tag{69}$$

This equation involves only the p's and q's, so it is a ϕ equation. It follows that we do not have the standard case. One can easily see that there are no other ϕ equations deducible from (68) alone, so the only other ϕ equations are (56), which now reads

$$\phi^r \equiv w^r + B_\mu A_\mu{}^r = 0, \tag{70}$$

and (57), which will be discussed later.

The field equations (58) give

$$(F_{\nu\mu})_\mu = m^2A_\nu. \tag{71}$$

We can infer by differentiating with respect to x_ν, using the condition $m \neq 0$,

$$A_{\nu\nu} = 0. \tag{72}$$

Also $$(F_{\nu\mu})_{-\mu} = -l_\mu l_\sigma(F_{\nu\mu})_\sigma + m^2A_\nu,$$

so that $$(l_\nu F_{\nu\mu})_{-\mu} = l_{\nu-\mu}F_{\nu\mu} + l_\nu(F_{\nu\mu})_{-\mu}$$
$$= m^2A_l$$

with the help of (12). Now from (68)

$$l_\nu F_{\nu\mu} = -B_\mu \Gamma^{-1} = -B_{-\mu}\Gamma^{-1},$$

so we get

$$(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l = 0. \tag{73}$$

This equation involves only the p's and q's, so it is a χ equation.

It is easily seen that B_l has zero P.b. with the first term of (73) and not with the second, provided $m \neq 0$. Thus B_l *must be a second class* ϕ *and* (73) *a second class* χ. There are thus no first class ϕ's besides ϕ^r and ϕ_l, so there are no extra terms in H besides the ones appearing in (32). Hence equation (62) holds strongly, i.e.

$$\mathfrak{H} \equiv B_\mu A_{\mu l} + \tfrac{1}{4} F_{\mu\nu}F_{\mu\nu}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu \Gamma. \tag{74}$$

We can now obtain $\mathfrak{H}$ as a function of the p's and q's by eliminating the normal derivative of A_μ from (74). We must take care to use only strong equations in this work, otherwise we may get extra terms containing B_l as a factor in the expression for $\mathfrak{H}$. With such extra terms in $\mathfrak{H}$, (57) would still be a correct ϕ equation, but its left-hand side would not be first class and would not be the ϕ_l occurring in H.

Let $F_{-\mu-\nu}$ be the tangential part of $F_{\mu\nu}$, given, according to a natural extension of (5), by

$$F_{-\mu-\nu} \equiv F_{\mu\nu} - l_\nu F_{\mu l} - l_\mu F_{l\nu} \tag{75}$$

where
$$F_{\mu l} \equiv -F_{l\mu} \equiv l_\sigma F_{\mu\sigma}.$$

Substituting
$$F_{\mu\nu} \equiv A_{\nu-\mu} - A_{\mu-\nu} + l_\mu A_{\nu l} - l_\nu A_{\mu l},$$
$$F_{\mu l} \equiv l_\sigma A_{\sigma-\mu} + l_\sigma l_\mu A_{\sigma l} - A_{\mu l}$$

in (75), we get after some reduction

$$F_{-\mu-\nu} \equiv A_{\nu-\mu} - A_{\mu-\nu} + l_\sigma(l_\mu A_{\sigma-\nu} - l_\nu A_{\sigma-\mu}). \tag{76}$$

Since the normal derivative of A_μ does not occur here, $F_{-\mu-\nu}$ is a function of p's and q's only.

We have

$$F_{-\mu-\nu}F_{-\mu-\nu} \equiv F_{\mu\nu}F_{-\mu-\nu} \equiv F_{\mu\nu}F_{\mu\nu} - 2F_{\mu l}F_{\mu l} \tag{77}$$

from (75). Thus (74) becomes

$$\mathfrak{H} \equiv B_\mu A_{\mu l} + \tfrac{1}{2} F_{-\mu-\nu}F_{-\mu-\nu}\Gamma + \tfrac{1}{2} F_{\mu l}F_{\mu l}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu \Gamma. \tag{78}$$

From (68) we get the strong equation

$$0 \equiv -\tfrac{1}{4}(B_\mu - F_{\mu l}\Gamma)(B_\mu - F_{\mu l}\Gamma)\Gamma^{-1}.$$

Adding this to (78), we get

$$\mathfrak{H} \equiv B_\mu l_\nu A_{\nu\mu} - \tfrac{1}{2} B_\mu B_\mu \Gamma^{-1} + \tfrac{1}{4} F_{-\mu-\nu}F_{-\mu-\nu}\Gamma - \tfrac{1}{2}m^2 A_\mu A_\mu \Gamma. \tag{79}$$

Equation (72) may be written

$$l_\nu A_{\nu l} + A_{\nu -\nu} = 0. \tag{80}$$

From it we can infer the strong equation

$$B_l(l_\nu A_{\nu l} + A_{\nu -\nu}) \approx 0$$

and hence

$$\begin{aligned} B_\mu l_\nu A_{\nu\mu} &\approx B_\mu l_\nu A_{\nu -\mu} + B_l l_\nu A_{\nu l} \\ &\approx B_{-\mu} l_\nu A_{\nu -\mu} - B_l A_{\nu -\nu}. \end{aligned}$$

Thus (79) becomes

$$\mathfrak{H} \approx B_{-\mu} l_\nu A_{\nu -\mu} - B_l A_{\nu -\nu} - \tfrac{1}{2} B_\mu B_\mu \Gamma^{-1} + \tfrac{1}{4} F_{-\mu -\nu} F_{-\mu -\nu} \Gamma - \tfrac{1}{2} m^2 A_\mu A_\mu \Gamma. \tag{81}$$

The normal derivative of A_μ has now disappeared and we have the correct expression for $\mathfrak{H}$ to use in (57).

To adapt the theory for quantization we must redefine P.b.'s by the method of section **8** of paper [2] so as to make the second class ϕ and χ equations (69) and (73) hold strongly. We take as the θ's of paper [2] the left-hand sides of (69) and (73). There are thus two θ's for each value of the u's, say

$$\theta \equiv (B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l, \qquad \theta^+ \equiv B_l. \tag{82}$$

Their P.b.'s are

$$[\theta,\theta'] = 0, \qquad [\theta^+,\theta^{+\prime}] = 0, \qquad [\theta,\theta^{+\prime}] = m^2\delta(u - u').$$

The coefficients c must be determined from equation (35) of paper [2], with the sum over s interpreted as a sum over θ and θ^+ together with an integral over all u values. The solution is that the c associated with $\theta(u)$ and $\theta^+(u')$ is $m^{-2}\delta(u - u')$ and the other c's vanish. Formula (36) of paper [2] then gives, with the sums over s and s' interpreted as above,

$$\begin{aligned} [\xi,\eta]^* = [\xi,\eta] &+ m^{-2}\int [\xi,(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l]\,[B_l,\eta]d^3u \\ &- m^{-2}\int [\xi,B_l]\,[(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l,\eta]d^3u. \end{aligned} \tag{83}$$

The new definition of P.b.'s makes B_l and $(B_{-\mu}\Gamma^{-1})_{-\mu} + m^2 A_l$ have zero P.b. with everything, so that one can put them strongly equal to zero without inconsistency. In working out the new P.b. of two given quantities ξ and η, it would be convenient first to make them independent of A_l and B_l by substituting

$$A_l \equiv -m^{-2}(B_{-\mu}\Gamma^{-1})_{-\mu}, \qquad B_l \equiv 0 \tag{84}$$

in them. If they are then independent of the w variables, we have $[\xi,B_l] = [\eta,B_l] = 0$, and so the new P.b. equals the old one. The formula (83) is thus needed only for evaluating the new P.b.'s of quantities involving the w variables.

When one passes to the quantum theory, the strong equations (84) become equations between operators. The weak ϕ equations (70) and (57) with $\mathfrak{H}$ given by (81) provide the Schrödinger wave equations, the dynamical variables in these equations having the new P.b. relationships.

Example 3. The electromagnetic field. We get the electromagnetic field by putting $m = 0$ in the vector meson field. This case has some special features which necessitate a separate investigation.

We now have

$$\mathfrak{L} \equiv -\tfrac{1}{4} F_{\mu\nu}F_{\mu\nu}. \tag{85}$$

Equations (67) ... (70) still hold and (69) is still a ϕ equation. Equation (71) becomes

$$(F_{\nu\mu})_\mu = 0. \tag{86}$$

Equation (72) cannot now be deduced. In the usual theory of electrodynamics, which was first given in a Hamiltonian form comparable with the present paper by Fermi [5], equation (72) is assumed as a supplementary condition. It will not be assumed in the present treatment.

Corresponding to (73), we now have the χ equation

$$\chi \equiv (B_{-\mu}\Gamma^{-1})_{-\mu} = 0. \tag{87}$$

It is easily seen that

$$[B_l,B'_l] = 0, \qquad [\chi,\chi'] = 0, \qquad [B_l,\chi'] = 0. \tag{88}$$

Since B_l and χ have zero P.b. with the other ϕ's, namely ϕ^r and ϕ_l, as can be inferred from ϕ^r and ϕ_l being first class, we see that B_l and χ must be first class. Thus *all the ϕ's and χ's are now first class.* This is the essential difference between the present example and the preceding one.

Let us see how to express H linearly in terms of the first class ϕ's. By using the analysis which led to (79), we can write (61) as

$$\begin{aligned} H &\equiv \int \dot{y}_l(w_l+l_{,\nu}A_\nu B_\mu-\tfrac{1}{2}B_\mu B_\mu\Gamma^{-1}+\tfrac{1}{4}F_{-\mu-\nu}F_{-\mu-\nu}\Gamma)d^3u+\int \dot{y}_r\phi^r d^3u \\ &\equiv \int \dot{y}_l\phi_l d^3u+\int \dot{y}_l l_{,\nu}A_\nu B_l d^3u+\int \dot{y}_r\phi^r d^3u, \end{aligned} \tag{89}$$

with ϕ_l defined by

$$\phi_l \equiv w_l + l_{,-\mu}B_{-\mu}- \tfrac{1}{2}B_\mu B_\mu\Gamma^{-1}+\tfrac{1}{4}F_{-\mu-\nu}F_{-\mu-\nu}\Gamma. \tag{90}$$

This expression for ϕ_l does not involve the normal derivative of A_μ, so it is a function of the p's and q's only, and it vanishes weakly, as it differs from the left-hand side of (55) only by a multiple of B_l, so it is a ϕ. It must be first class, as all the ϕ's are now first class. Thus (89) expresses H as a linear function of first class ϕ's.

The ϕ_l introduced above may be considered as the ϕ which gives rise to the motion of the surface normal to itself. However, we could take an alternative

ϕ_l, differing from (90) by any function of the p's and q's which vanishes weakly through being a linear function of B_l, and could consider it equally well as the ϕ which gives rise to the motion of the surface normal to itself. An example of special interest will here be given.

Putting A_l for V in (49) and writing $A_l{}^\tau$ for $dA_l/d\tau$ to have an unambiguous notation, we get

$$A_l{}^\tau - y_{,r}A_l{}^r \equiv y_l A_{11} \equiv y_l(l_{,r}A_{,l} + l_{,l}A_{,r}). \tag{91}$$

Putting $y_{,}{}^\tau$ for V in the same formula, we get

$$y_l y_{,}{}^\tau{}_l \equiv y_{,}{}^\tau - y_{,s}y_{,}{}^{\tau s}.$$

Multiplying this by $l_{,}$, we get

$$-y_l y_{,}{}^\tau l_{,l} \equiv y_{,}{}^\tau l_{,} + y_{,s}y_{,}{}^s l_{,}{}^\tau \equiv y_l{}^\tau.$$

Multiplying again by $A_{,}$ and subtracting (91), we get

$$-A_l{}^\tau + y_{,r}A_l{}^r \equiv y_l{}^\tau A_{,} - y_{,l}A_{,l}.$$

With the help of this result the second term of (89) becomes

$$\begin{aligned}\int y_l l_{,r}A_{,l}B_l d^3u &\equiv \int (A_l{}^\tau - y_{,r}A_l{}^r + y_l{}^\tau A_{,})B_l d^3u \\ &\equiv \int A_l{}^\tau B_l d^3u - \int y_{,r}A_l{}^r B_l d^3u - \int y_l(A_{,}B_l)^\tau d^3u.\end{aligned}$$

So we may write (89) as

$$H \equiv \int y_l \phi^+{}_l d^3u + \int A_l{}^\tau B_l d^3u + \int y_{,r}\phi^{+r\tau} d^3u, \tag{92}$$

with

$$\phi^+{}_l \equiv \phi_l - (A_{,}B_l)^r \tag{93}$$

and

$$\phi^{+r\tau} \equiv \phi^r - A_l{}^\tau B_l. \tag{94}$$

We now have H expressed in terms of the first class ϕ's $\phi^+{}_l$, B_l, $\phi^{+r\tau}$. We may look upon $\phi^+{}_l$ as an alternative ϕ_l giving rise to the motion of the surface normal to itself and $\phi^{+r\tau}$ as an alternative ϕ^r giving rise to a change of parametrization. It should be noted that $\phi^+{}_l$ and $\phi^{+r\tau}$ have zero P.b. with A_l. We have from (67)

$$[A_l, A'_l{}^{\tau'} B'_l] = A_l{}^\tau \delta(u - u'),$$

and since A_l is a u-scalar it satisfies the condition (38), which gives

$$[A_l, \phi'^{\tau}] = A_l{}^\tau \delta(u - u'),$$

and hence

$$[A_l, \phi^{+r\tau}] = 0. \tag{95}$$

Again, with the help of (21)

$$(96) \qquad [A_l,\phi^{+\prime}{}_l] = A_\lambda[l_\lambda,w'_l] - l_\lambda[A_\lambda,(A'_r,B'_l)^{r'}]$$
$$= -A_\lambda\delta_{-\lambda}(u-u') - l_\lambda\{A'_r,l'_\lambda\delta(u-u')\}^{r'} = 0.$$

The Hamiltonian contains an extra term besides those that give rise to arbitrary changes in the surface and its parametrization, namely the second term in (89) or (92). This extra term gives a further freedom in the motion. It allows A_l to vary arbitrarily with τ, the equation $A_l{}^\tau = [A_l,H]$ being identically fulfilled, as follows from the expression (92) for H, with the help of (95) and (96).

The further freedom corresponds physically to the possibility of changes of gauge taking place while the motion develops. The initial conditions, fixing an initial surface and the potentials and their normal derivatives on it, do not restrict the gauge at points in space-time away from this surface. One can make a gauge transformation

$$(97) \qquad A_\mu \rightarrow A_\mu + \partial S/\partial x_\mu$$

with S an arbitrary function of the four x_μ's. Thus one can choose S so that there is no change in the conditions on the initial surface while there is an arbitrary change of gauge in other regions of space-time. This change will affect the dynamical variables at later τ values and give rise to arbitrary functions in the solution of the equations of motion, even when the motion of the surface is prescribed.

In the usual theory of electrodynamics one has the supplementary condition (72), which results in the S of (97) being restricted to satisfy

$$(98) \qquad S_{\mu\mu} = 0.$$

One can then no longer make a change of gauge without affecting the potentials or their normal derivatives on the initial surface, so the extra arbtrariness in the motion no longer occurs. The present theory of electrodynamics differs from the usual one through allowing more general gauge transformations, but the two theories are equivalent for all gauge invariant effects, and thus for all effects of physical importance.

The question arises with the present theory whether one can have a motion for which the gauge changes while the surface and its parametrization do not change. Using the form (92) for H, it is immediately evident that one can have such a motion, since one can put $\dot{y}_l = \dot{y}_r = 0$ in this expression for H and the second term survives, leaving the rate of change of A_l arbitrary. A general change of gauge involves independent changes in the normal component of A_μ, namely A_l, and the three-dimensional divergence of its tangential component, namely $A_{-\mu-\mu}$, on the surface. Thus the change of gauge allowed by our equations of motion when there is no change in the surface is not a general one.

If we require the trajectories of the motion in phase space to form integrable subspaces, in the way discussed on page 142 of paper [2], we must be able to have a general change of gauge with no change in the surface, since such a change could be attained by first moving the surface and making some change in the gauge, and then moving the surface back again and making a further change of gauge, not cancelling the previous one in any way. In order to get equations of motion allowing general changes of gauge with no change in the surface, we must add a further term to H, namely

$$\int v\chi d^3u \tag{99}$$

with χ given by (87) and the coefficient v arbitrary. This means treating χ as a first class ϕ. The Hamiltonian modified in this way is not derivable from an action density, but is still a permissible Hamiltonian for a dynamical system, leading to consistent equations of motion, on account of χ being first class. The modification in H merely adds to the solutions of the equations of motion without altering the previously existing solutions, the latter being just the special case $v = 0$ of the new solutions. Thus one can consider the modification as not a change to a new dynamical system, but merely an extension of the treatment of the original dynamical system.

We can now pass to the quantum theory by making each first class ϕ, including the first class χ's that get changed into first class ϕ's to satisfy the integrability condition, into a Schrödinger wave equation. Thus we get the wave equations

$$\phi^+{}_l\psi = 0, \qquad \phi^{+r}\psi = 0, \qquad B_l\psi = 0, \qquad (B_{-s}\Gamma^{-1})_{-r}\psi = 0. \tag{100}$$

The last two of these equations show that the wave function ψ, if expressed in terms of the longitudinal and transverse components of the A's on the surface, is independent of the longitudinal components. It thus involves the longitudinal field variables in a different way from the usual quantum electrodynamics.

References

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[2] P. A. M. Dirac, *Generalized Hamiltonian dynamics*, Can. J. Math., vol. 2 (1950), 129.

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AXIOMATIQUE DES LATTICES DISTRIBUTIVES

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CETTE note a pour principal objet de répondre à deux questions posées par G. Birkhoff.[1]

§1

THÉORÈME 1. *Les lattices distributives avec O et I sont exactement les systèmes algébriques avec une opération ternaire* (a, b, c) *et des éléments* O *et* I *tels que*:

$$(1.1) \qquad (O, a, I) = a,$$

$$(1.2) \qquad (a, b, a) = a,$$

$$(1.3) \qquad (d, (a, b, c), e) = (b, (c, d, e), (a, d, e)).$$

(*On définit* $a \cup b = (a, I, b)$ *et* $a \cap b = (a, O, b)$. *Alors* $(a, b, c) = (a \cap b) \cup (b \cap c) \cup (c \cap a)$.) *De plus, les axiomes* (1.1), (1.2), (1.3) *sont indépendants.*

Je ramène la démonstration à celle du théorème 4, p. 137.[1] Les axiomes du théorème 4 entraînent évidemment les précédents. Etablissons la réciproque. On a successivement:

Par (1.2), (1.3), (1.2),

$$(1.4) \qquad a = (a, (a, b, a), a) = (b, (a, a, a), (a, a, a)) = (b, a, a).$$

Par (1.4), (1.3), (1.2),

$$(1.5) \qquad (a, d, e) = (b, (a, d, e), (a, d, e)) = (d, (a, b, a), e) = (d, a, e).$$

Par (1.1), (1.3), (1.5), (1.1),

$$(1.6) \qquad \begin{aligned}(a, b, c) &= (O, (a, b, c), I) = (b, (c, O, I), (a, O, I)) \\ &= (b, (O, c, I), (O, a, I)) = (b, c, a).\end{aligned}$$

L'axiome (17) du théorème 4 résulte alors directement de (1.3), (1.5), (1.6). L'indépendance des axiomes est prouvée par les contre-exemples suivants:

$E_1 = \{O, I\}$ avec $(a, b, c) = c$.

$E_2 = \{O, x, I\}$ avec $(a, b, c) = x$ si au moins un des trois éléments a, b, c est x; sinon, $(a, b, c) = O$ ou I selon que au moins deux ou moins de deux des trois éléments sont O.

Reçu le 26 janvier, 1950.

[1] G. Birkhoff, *Lattice theory*, (2ième ed.) 1949, Amer. Math. Soc. Colloquium publications, vol. XXV. Problèmes 64 et 65, pp. 138 et 139. Pendant l'impression de cette note, Ph. Vassiliou m'a communiqué une solution voisine du problème 64. Cette solution est publiée par l'Université technique nationale d'Athènes, nº 5, 1950.

$E_3 = \{O, I\}$ avec $(I, I, O) = O$; $(I, O, O) = I$; sinon, $(a, b, c) = O$ ou I comme dans E_2.

§2

Par la même méthode, on déduit du système de 5 postulats des algèbres de Boole de Grau[2] un système de 2 postulats (indépendants).

D'après Grau, un système algébrique avec une opération ternaire (a, b, c) dans lequel, pour tout a, existe au moins un complément a', avec les propriétés:

$$((a, b, c), d, e) = ((a, d, e), b, (c, d, e)),$$
$$(b, b, a) = (a, b, b) = b,$$
$$(b', b, a) = (a, b, b') = a,$$

peut être organisé en algèbre de Boole (et réciproquement).

Je remplace ces postulats par les suivants:

(2.1) $$(d, e, (b, a, c)) = (b, (e, d, c), (d, e, a)),$$

(2.2) $$(b, a, b') = a.$$

D'après les résultats de Grau, ce système est conséquence du précèdent. La réciproque est vraie; on a successivement:

Par (2.2), (2.2), (2.1), (2.2),

(2.3) $$\begin{aligned} a &= (b, a, b') = (b, a, (a, b', a')) \\ &= (a, (a, b, a'), (b, a, b')) = (a, b, a). \end{aligned}$$

Par (2.3), (2.1), (2.3),

(2.4) $$\begin{aligned} (b, a, a) = (b, a, (a, b, a)) &= (a, (a, b, a), (b, a, b)) \\ &= (a, a, b). \end{aligned}$$

Par (2.3), (2.1), (2.4), (2.1), (2.3), (2.3),

(2.5) $$\begin{aligned} (b, a, a) &= (b, (a, a, a), (a, a, a)) = (a, a, (b, a, a)) \\ &= (a, a, (a, a, b)) = (a, (a, a, b), (a, a, a)) \\ &= (a, (a, a, b), a) = a. \end{aligned}$$

Par (2.5), (2.1), (2.4), (2.5),

(2.6) $$(a, b, c) = (a, b, (x, c, c)) = (x, (b, a, c), (a, b, c)) = (b, a, c),$$

si on prend $x = (b, a, c)$.

Par (2.2), (2.2), (2.6), (2.1), (2.6), (2.2), (2.3),

(2.7) $$\begin{aligned} a = (b, a, b') &= (b, a, (b, b', b')) = (b, a, (b', b, b')) \\ &= (b', (a, b, b'), (b, a, b)) = (b', (b, a, b'), (b, a, b)) \\ &= (b', a, b). \end{aligned}$$

[2] A. A. Grau, *Ternary Boolean Algebra*, Bull. Amer. Math. Soc., vol. 53 (1947), pp. 567-572.

Par (2.2), (2.7), (2.1), (2.5), (2.2),

$$(2.8)\qquad a = (b, a, b') = (b, a, (b', b', b)) = (b', (a, b, b), (b, a, b')) = (b', b, a).$$

Par (2.8), (2.1), (2.6), (2.8),

$$(2.9)\qquad (a, b, c) = (x', x, (a, b, c)) = (a, (x, x', c), (x', x, b)) = (a, c, b).$$

De (2.6) et (2.9), on déduit facilement tous les axiomes de Grau.

Les axiomes (2.1) et (2.2) sont indépendants:

$E_1 = \{O, I\}$ avec $(x, y, z) = y$.
$E_2 = \{O, I\}$ avec $(x, y, z) = O$.

§3

En ce qui concerne le problème 65 de Birkhoff, j'établis que les postulats du théorème 3 sont indépendants, puis je montre qu'une modification convenable de l'un d'eux permet d'en supprimer trois autres. Le théorème 3 peut s'énoncer ainsi:

Un système algébrique qui satisfait les axiomes (S):

$$(3.1)\qquad a \cap a = a,$$

$$(3.2\alpha)\qquad a \cup I = I \quad \text{pour un } I,$$

$$(3.2\beta)\qquad I' \cup a = I' \quad \text{pour un } I',$$

$$(3.3\alpha)\qquad a \cap J = a \quad \text{pour un } J,$$

$$(3.3\beta)\qquad J' \cap a = a \quad \text{pour un } J'$$

(on voit de suite que $I = I'$ et $J = J'$),

$$(3.2\text{–}3.3)\qquad I = J,$$

$$(3.4\alpha)\qquad a \cap (b \cup c) = (a \cap b) \cup (a \cap c),$$

$$(3.4\beta)\qquad (b \cup c) \cap a = (b \cap a) \cup (c \cap a)$$

est une lattice distributive avec I.

THÉORÈME II. *Les axiomes précédents sont indépendants.*

E_1:

$\cap$	a	b	I
a	a	a	a
b	a	a	b
I	a	b	I

$\cup$	a	b	I
a	a	b	I
b	b	b	I
I	I	I	I

$E_{4\alpha}$:

$\cap$	a	b	I
a	a	b	a
b	a	b	b
I	a	b	I

$\cup$	a	b	I
a	a	b	I
b	a	b	I
I	I	I	I

$E_{2\alpha}$:

$\cap$	a	I
a	a	a
I	a	I

$\cup$	a	I
a	a	a
I	I	I

$E_{3\alpha}$:

$\cap$	a	I
a	a	I
I	a	I

$\cup$	a	I
a	a	I
I	I	I

E_{2-3}:

$\cap$	I	J
I	I	I
J	I	J

$\cup$	I	J
I	I	I
J	J	I

$E_{2\beta}$, $E_{3\beta}$, $E_{4\beta}$: anti-isomorphes de $E_{2\alpha}$, $E_{3\alpha}$, $E_{4\alpha}$.

THÉORÈME III. *Le système d'axiomes* (S) *est équivalent au système* (S') *d'axiomes également indépendants*:

(3.1), (3.2α), (3.3α), (3.2–3.3) et

$$(3.4') \qquad a \cap (b \cup c) = (c \cap a) \cup (b \cap a).$$

Il suffit d'établir que (S') entraîne (S). On établit successivement:

Par (3.3α), (3.2–3.3), (3.2α), (3.4'),

$$(3.5) \qquad a = a \cap (I \cup I) = (I \cap a) \cup (I \cap a).$$

Par (3.4'), (3.3α),

$$(3.6) \qquad I \cap (a \cup a) = (a \cap I) \cup (a \cap I) = a \cup a.$$

Par (3.5), (3.6), (3.5),

$$(3.3\beta) \qquad I \cap a = I \cap [(I \cap a) \cup (I \cap a)] = (I \cap a) \cup (I \cap a) = a.$$

Par (3.3α), (3.2α), (3.4'), (3.3β), (3.1),

$$(3.7) \qquad a = a \cap I = a \cap (a \cup I) = (I \cap a) \cup (a \cap a) = a \cup a.$$

Par (3.7), (3.4'), (3.7),

$$(3.8) \qquad a \cap b = a \cap (b \cup b) = (b \cap a) \cup (b \cap a) = b \cap a.$$

Par (3.3β), (3.4'), (3.3α),

$$(3.9) \qquad b \cup c = I \cap (b \cup c) = (c \cap I) \cup (b \cap I) = c \cup b.$$

Alors, (3.2β) résulte de (3.9); (3.4α) et (3.4β) résultent de (3.4'), (3.8), (3.9).

Les contre-exemples suivants montrent l'indépendance:

E'_1 et E'_{2-3}: E_1 et E_{2-3}.

$E'_{2\alpha}$:

$\cap$	a	I
a	a	a
I	a	I

$\cup$	a	I
a	a	a
I	a	a

E'_3:

$\cap$	a	I
a	a	I
I	a	I

$\cup$	a	I
a	I	I
I	I	I

E'_4:

$\cap$	a	I
a	a	I
I	I	I

$\cup$	a	I
a	I	I
I	I	I

Tous les contre-exemples utilisés sont tels qu'il n'en existe pas avec un nombre moindre d'éléments. Pour E_1 et $E_{4\alpha}$, il n'existe pas de contre-exemple avec 2 éléments seulement.

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POSTULATES FOR DISTRIBUTIVE LATTICES

MARLOW SHOLANDER

MANY sets of postulates have been given for distributive lattices and for Boolean algebra. For a description of some of the most interesting and for references to others the reader is referred to Birkhoff's "Lattice Theory"[1]. In this paper we give sets of postulates which have some intrinsic interest because of their simplicity. In the first two sections binary operations are used to describe a distributive lattice by 2 identities in 3 variables and a Boolean algebra by 3 identities in 3 variables. In the third section a ternary operation is used to describe distributive lattices with O and I by 2 identities in 5 variables.

1. Distributive lattices. Let $\mathfrak{S}$ be a set of elements $a, b, c, \ldots$ closed under the operations $\cup$ and $\cap$ and satisfying, for all a, b, c in $\mathfrak{S}$, these postulates:

P1. $$a = a \cap (a \cup b),$$

P2. $$a \cap (b \cup c) = (c \cap a) \cup (b \cap a).$$

We wish to prove $\mathfrak{S}$ is a distributive lattice. In identities (1.1), (1.2), and (1.3) below, A denotes $a \cap a$.

(1.1) $$a = a \cap (a \cup a) = A \cup A,$$ by P1 and P2.

(1.2) $$a = a \cap a.$$

Proof. $$A = A \cap (A \cup A) = A \cap a,$$ by P1 and (1.1).

Hence
$$\begin{aligned} a \cap a &= a \cap (A \cup A) \\ &= (A \cap a) \cup (A \cap a) \\ &= A \cup A = a, \end{aligned}$$ by (1.1), P2, and (1.1).

(1.3) $$a = A \cup A = a \cup a,$$ by (1.1) and (1.2).

(1.4) $$a \cap b = b \cap a.$$

Proof.
$$\begin{aligned} a \cap b &= (a \cap b) \cup (a \cap b) \\ &= b \cap (a \cup a) = b \cap a, \end{aligned}$$ by (1.3), P2, and (1.3).

(1.5) $$a = (b \cap a) \cup a.$$

Proof.
$$\begin{aligned} a &= a \cap (a \cup b) \\ &= (b \cap a) \cup a, \end{aligned}$$ by P1, P2, and (1.2).

(1.6) $$a = a \cup [(b \cap a) \cap a].$$

Proof.
$$\begin{aligned} a &= a \cap a = a \cap [(b \cap a) \cup a] \\ &= a \cup [(b \cap a) \cap a], \end{aligned}$$ by (1.2), (1.5), P2, and (1.2).

Received June 20, 1950. This paper was written while the author was under contract to the Office of Naval Research.

(1.7) $a \cup b = [b \cup (a \cap b)] \cup a.$

Proof.
$$\begin{aligned} a \cup b &= (a \cup b) \cap (a \cup b) \\ &= [b \cap (a \cup b)] \cup [a \cap (a \cup b)] \\ &= [b \cup (a \cap b)] \cup a, \qquad \text{by (1.2), P2, P2, (1.2) and P1.} \end{aligned}$$

(1.8) $b = b \cup (a \cap b).$

Proof.
$$\begin{aligned} b &= (a \cap b) \cup b \\ &= \{b \cup [(a \cap b) \cap b]\} \cup (a \cap b) \\ &= b \cup (a \cap b), \qquad \text{by (1.5), (1.7), and (1.6).} \end{aligned}$$

(1.9) $a \cup b = b \cup a,$ by (1.7) and (1.8).

Since the remainder of the exposition has a pattern common to several previous expositions [1, pp. 135, 136.], we proceed giving somewhat less detail. We have proved the so-called idempotent, commutative, and absorption laws. The associative laws remain to be proved.

We denote $(a \cup b) \cup c$ by P and $a \cup (b \cup c)$ by Q. It is routine to show that $a \cap P = a$, $b \cap P = b$, and $c \cap P = c$. Hence,

$$\begin{aligned} Q &= (a \cap P) \cup [(b \cap P) \cup (c \cap P]) \\ &= (a \cap P) \cup [(b \cup c) \cap P] = Q \cap P. \end{aligned}$$

By left-right symmetry, $Q \cap P = P$. Thus we have $\cup$ associativity and it is now easy to deduce the dual of the distributive law. By duals of proofs previously given, we may prove $\cap$ associativity. We then have

(1.10) $\mathfrak{S}$ is a distributive lattice.

2. Distributive lattices with O and I. In this section we note some immediate extensions of the postulate system P1, P2. Consider the postulates:

P3. $a \cup O = a,$ for some O.

P3′. $a \cap I = a,$ for some I.

P3″. $O \cup (a \cap I) = a,$ for some O and some I.

P3*. To each b there corresponds some b' such that

$$a \cap (b \cup b') = a \cup (b \cap b').$$

Using (1.10) it is easy to prove the following statements. An algebraic system which satisfies P1, P2 and

(i) P3, is a distributive lattice with O,

(ii) P3′, is a distributive lattice with I,

(iii) P3″, is a distributive lattice with O and I,

(iv) P3*, is a Boolean algebra.

In case (i), we have $a \cap O = (a \cup O) \cap O = O$. Moreover O is unique for if an element O' shares the properties of O, then $O = O' \cup O = O'$. Case (ii) is the dual of case (i).

In case (iii), we have $O \cup a = O \cup [O \cup (a \cap I)] = O \cup (a \cap I) = a$ and hence $a = O \cup (a \cap I) = a \cap I$. Thus P3'' implies P3 and P3'.

In case (iv), denote $b \cup b'$ by I and $b \cap b'$ by O. From $a \cap I = a \cup O$, $O \cup (a \cap I) = O \cup (a \cup O) = a \cup O = a \cup (a \cup O) = a \cup (a \cap I) = a$. Hence P3* implies P3''. It is a routine matter to show that the complement, b', of b is unique.

3. Postulates with a ternary operation. The ternary operation used here is the one introduced by Grau [3]. Kiss and Birkhoff [4] have described distributive lattices with O and I in terms of the operation. Croisot [2], using this operation and 5 variables, defines a Boolean algebra by means of 2 identities and a distributive lattice with O and I by means of 3 identities (see Problem 64 in [1]). In the latter case also, it happens that 2 identities are sufficient. We give the result without proof.

Let $\mathfrak{S}$ be an algebraic system with a ternary operation (a, b, c) and with elements O and I such that, identically,

Q1. $$(O, a, (I, b, I)) = a,$$

Q2. $$(a, (b, c, d), e) = ((a, c, e), d, (b, a, e)).$$

If we define $a \cup b = (a, I, b)$ and $a \cap b = (a, O, b)$, then S is a distributive lattice with O and I. Moreover,

$$(a, b, c) = (a \cap b) \cup (b \cap c) \cup (c \cap a).$$

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SHORT FORMULATIONS OF BOOLEAN ALGEBRA, USING RING OPERATIONS

LEE BYRNE

SPECIAL interest has recently attached to formulations of Boolean algebra in terms of ring operations [7], [1]. These axiomatizations have not been as brief as those reached through other modes of approach.[1]

The present note will show that the number of axioms when ring operations are used may be as small as in any present version that is not metamathematical,[2] that is, the number of axioms finally employed will be two.[3]

The first step will be to give a set of four postulates of more familiar appearance; these will mark a reduction by about two, as compared with similar versions of earlier date. Then, by employing a device due to Bernstein [2], we shall set up two postulates from which the previous four can be deduced.

FORMULATION IN FOUR TRANSFORMATION AXIOMS

Axioms

I. $(X + Y) + Z = X + (Y + Z)$. II. $(X + Y) + X = Y$.

III. $X(W + YZ) = XW + Y(ZX)$. IV. $X(X + 1) = Y + Y1$.

Theorems[4]

1. $X + Y = Y + X$.

Proof. $((X + Y) + (X + Y)) + (X + Y)$
$= ((X + Y) + X) + ((Y + X) + Y)$. From I, I.

The Theorem follows by II, II, II.

2. $(X + X) + Y = Y = Y + (X + X)$.

Proof. $Y = (X + Y) + X = (Y + X) + X$
$= Y + (X + X) = (X + X) + Y$. From II, 1, I, 1.

2a. $X + Y = Z + Z \rightarrow X = Y$.

Proof. $X + Y = Z + Z \rightarrow (X + Y) + Y$
$= X + (Y + Y) = X = (Z + Z) + Y = Y$. From I, 2, 2.

3. $X + X = Y + Y$.

Proof. $X + X = (X + X) + (Y + Y) = Y + Y$. From 2, 2.

Received November 10, 1949.

[1] References to several brief versions are given in [4]; see also [5].

[2] An interesting metamathematical version [6] uses only *one* transformation postulate.

[3] More precisely, there will be two "transformation" axioms, and these will not include the tacit assumptions as to closure with respect to operations or as to a minimal number of elements.

[4] An arrow $\rightarrow$ signifies (material) implication, as in Hilbert and Ackermann.

4. $X(Y + Y) = Z + Z$.
Proof. $X(XX + XX) = X(XX) + X(XX)$. From III.
The Theorem follows by 3.

5. $X(YZ) = Y(ZX) = Z(XY)$.
Proof. $X(YZ + YZ) = W + W = X(YZ) + Y(ZX)$. From 4, III.
The Theorem follows by 2a.

6. $X1 = X$.
Proof. $1(1 + 1) = X + X1$. From IV.
The Theorem follows by 4, 2a.

6a. $1X = X$.
Proof. $1(X1) = 1(X) = X(11) = X1 = X$. From 6, 5, 6, 6.

7. $XY = YX$.
Proof. $X(Y1) = XY = Y(1X) = YX$. From 6, 5, 6a.

7a. $X(YZ) = (XY)Z$.
Proof. $X(YZ) = Z(XY) = (XY)Z$. From 5, 7.

8. $X(Y + Z) = XY + XZ$.
Proof. $X(Y + 1Z) = XY + 1(ZX) = XY + 1(XZ)$. From III, 7.
The Theorem follows by 6a.

8a. $(X + Y)Z = XZ + ZY$.
Proof. $Z(X + Y) = ZX + ZY$. From 8.
The Theorem follows by 7.

9. $XX = X$.
Proof. $X(X + 1) = XX + X = Y + Y1 = Y + Y$. From IV, 8, 6.
The Theorem follows by 2a.

With the tacit assumptions (Note 3), I, 1, 2, 3, 5a, 7, 8, 8a show the system to be a ring, 9 Boolean, 6, 6a with unit, and accordingly a Boolean algebra [3, pp. 96, 97] and it is known that the axioms used are themselves valid in Boolean algebra. Note that $X + X$ is the zero, and Theorem 3 shows both unicity of zero and also that every element is its own negative. The complement of any X is $X + 1$.

Formulation in Two Transformation Axioms
(using a device of Bernstein[2])

Abbreviations: $P_1 = ((A + B) + C + (A + (B + C))$
$P_2 = C(F + DE) + (CF + D(EC))$
$P_3 = G(G + 1) + (H + H1)$
$0 = 1 + 1$

Axioms

I′. $X + X = P_1 + (P_2 + P_3)$. II. $(X + Y) + X = Y$.

Theorems[5]

3. $X + X = Y + Y = 1 + 1$. From I'.

2. $(X + X) + Y = Y = Y + (X + X)$.

Proof. $((Y + Y) + Y) + (Y + Y) = Y$
$= (Y + Y) + Y = Y + (Y + Y)$. From II, II, II.

The Theorem follows by 3.

2a. $X + Y = Z + Z \to X = Y$.

Proof. $X + Y = Z + Z \to (X + Y) + X = (Z + Z) + X$.

The Theorem follows by II, 2.

11. $P_1 = P_2 + P_3$. From I' by 2a.

12. $P_2 = P_3$.

Proof. $((C + C) + C) + (C + (C + C)) = P_2 + P_3$. From 11.

The Theorem follows by 2, 2a.

I. $(X + Y) + Z = X + (Y + Z)$. From 11 by 12, 2a.

III. $X(W + YZ) = XW + Y(ZX)$.

Proof. $P_2 = 0(0 + 1) + (0 + 01) = 01 + 01$. From 12, 2.

The Theorem follows by 2a.

IV. $X(X + 1) = Y + Y1$. From 12 by III 2a.

Thus by means of two valid axioms we have derived the previous four[6].

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[5] Numbered as before, with 11 and 12 as new theorems.

[6] If it is desired to formalize also the tacit assumptions mentioned in Note 3, we may append A. $X_1 \neq X + 1$, and the closure postulate B. $(X)\,(Y)\,(Z)\,(EV)\,(EW)\,V = X + Y$ & $W = VZ$), where (X) is "for every X" and (EZ) "there exists a Z such that". Then (after obtaining 2) we can deduce as Theorems:

01. $(X)\,(Y)\,(EZ)\,(Z = X + Y)$; from B.

02. $(X)\,(Y)\,(EZ)\,(Z = XY)$: since $(X)\,(Y)\,(EW)\,(EZ)\,(W = X + (V + V)\ \&\ Z = WY)$, from B,B, whence Theorem by 2.

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THE IMMERSIBILITY OF A SEMIGROUP INTO A GROUP

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1. Introduction. A *semigroup* is a set of elements which is closed under an associative operation, usually called multiplication. When can a semigroup be embedded in a group, i.e., under what condition is it isomorphic to a subset of a group? A necessary condition for immersibility is clearly the so-called *cancellation law*:

(C) *If* $ax = ay$ *or* $xb = yb$, *then* $x = y$.

It is well known that a finite semigroup with cancellation law is a group, also that an Abelian semigroup (one in which multiplication is commutative) can be embedded in a group if and only if the cancellation law holds. In general however the cancellation law is not sufficient for immersibility, as was shown by A. Malcev in 1936.

In a second paper of 1939, Malcev stated necessary and sufficient conditions. They assert that certain chains of equations imply further equations. The number of these conditions is infinite, and the chains are of unbounded length. He proved in a third paper of 1940 that no finite part of these conditions will suffice.

Malcev offers a rather complicated construction for obtaining such chains of equations. In the present paper I have tried to give a simpler construction, by the device of using parts of polyhedra, rather than natural numbers, for labelling the equations and variables contained in these conditions. Acquaintance with Malcev's work will not be expected from the reader. In defining the term "polyhedron", I shall roughly follow the book on topology by Seifert and Threlfall.

A *face* (abstract polygon) is a division of a topological circle into two or more arcs, called *sides*, by an equal number of points, which we will term *angles*. An abstract polyhedron is a system of F faces, containing together $2E$ sides, such that every side is mapped topologically on exactly one other. A pair of sides thus mapped into each other is called an *edge*. Hence every edge has two sides. We may speak about the *midpoint* (some interior point) of an edge, which divides the edge into two *half-edges*. A set of angles which correspond to one another under the mapping is called a *vertex*. Every edge has two vertices. To every edge there belong four angles, which may be classified by pairs in two different ways: angles at the same vertex of the edge, and angles on the same side of the edge. Every half-edge has one vertex, two sides, and two angles. The polyhedron is called *Eulerian* if the total number of

Received February 22, 1950.

vertices is V such that $V + F - E = 2$. This is also a necessary and sufficient condition for the surface defined by the polyhedron to be homeomorphic (i.e. topologically equivalent) to the sphere. Throughout the present paper we shall always mean "abstract Eulerian polyhedron" when we say "polyhedron."

Given a semigroup $\mathfrak{H}$, we shall understand by *polyhedral condition* the following statement:

(P) *If the elements of $\mathfrak{H}$ are assigned to all sides and angles of any Eulerian polyhedron, such that to each half-edge there corresponds an equation $xa = yb$, where x and y have been assigned to the sides, a and b to the corresponding angles of the half-edge, then these $2E$ equations are interdependent, i.e. any one of them can be derived from the totality of all others.* (See Figure 1.)

The application of this condition to two polyhedra which are topologically equivalent will of course give the same result. We shall prove that (P) is a necessary and sufficient condition for immersibility of a semigroup $\mathfrak{H}$ with cancellation law into a group. This will establish the following

THEOREM: *A semigroup can be embedded in a group if and only if the cancellation law* (C) *and the polyhedral condition* (P) *are satisfied.*

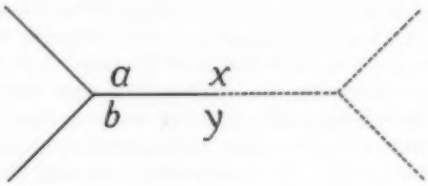

Fig. 1

2. Necessity of polyhedral condition. Let the semigroup $\mathfrak{H}$ be contained in a group $\mathfrak{G}$, so that the elements of $\mathfrak{H}$ possess inverses in $\mathfrak{G}$. Assign elements of $\mathfrak{H}$ to the sides and angles of a given polyhedron. Assume that the equations belonging to all but one half-edge are true, the remaining equation is to be deduced.

An *oriented triangulation* of the polyhedron is obtained as follows: Directed radii are drawn from the centre (some interior point) of each face to the angles of the face. (It will be remembered that faces are circles, and that by angles we understand points on the circumference.) Directed radii are drawn from the midpoints of all sides to the centre of the face. Each half-edge of the original polyhedron is given a direction from the midpoint of the edge towards the vertex. The half-edges thus oriented as well as the directed radii will be the oriented edges of the triangulation.

The equation $xa = yb$, corresponding to any half-edge of the polyhedron, can be replaced by the two equations $xa = p$ and $yb = p$, corresponding to

two triangles of the triangulation. The variables occurring in these equations are assigned to the edges of the triangulation: namely x and y to the radii from the midpoint, a and b to the radii towards the vertex, and p to the half-edge itself. (See Fig. 2.)

If the equation $xa = yb$ was to be inferred, we may now add $yb = p$ to the given equations, and leave only $xa = p$ to be deduced. Thus, corresponding to each oriented triangle, we have an equation; for instance $xa = p$ and $yb = p$ correspond to two of the four triangles on Fig. 2. Of these $4E$ equations all but one are given, and one is to be derived.

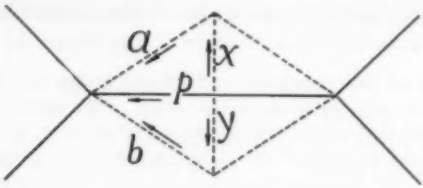

Fig. 2

Consider any path made up entirely of edges of the triangulation. Corresponding to such a path we form a product in the following way: If the element x of $\mathfrak{H}$ has been assigned to the nth edge of the path, then the nth term of the product is x or x^{-1}, depending on whether this edge has been traversed in the right or in the wrong direction. For instance, there are six different closed paths by means of which the perimeter of the upper left triangle of Fig. 2 can be once described. Correspondingly we may obtain one of the six products:

$$(\dagger) \qquad xap^{-1},\ ap^{-1}x,\ p^{-1}xa;\quad pa^{-1}x^{-1},\ a^{-1}x^{-1}p,\ x^{-1}pa^{-1}.$$

As long as $xa = p$, each of these products has the value 1: and conversely, if any one of the six products (†) is 1, then $xa = p$.

Consider now a closed path consisting of edges of the triangulation, and surrounding only triangles for which the corresponding equations are given. We prove by induction that the product corresponding to this path will be unity. We have shown above that this is indeed so, if the path surrounds only one triangle. Otherwise we may decompose the closed path into two paths Q and R traversed in succession such that there will exist a path P, lying entirely inside the closed path, and joining the endpoint of Q to the endpoint of R. Let P' be the path P traversed in opposite direction. If $f(P)$ denotes the product associated with P, then $f(P)\,f(P') = 1$. Hence $f(Q)\,f(R) = f(Q)\,f(P)\,f(P')\,f(R) = 1$, since $f(Q)\,f(P) = f(P')\,f(R) = 1$, by induction hypothesis.

Suppose now the upper left triangle of Fig. 2 is the one for which the corresponding equation is to be derived. Since the surface defined by our poly-

hedron is homeomorphic to the sphere, the perimeter of this triangle divides the whole surface into two simply connected regions. Let us describe a closed path along this perimeter, and call the outside of the triangle the inside of the path. Corresponding to this path we obtain one of the six products (†), whose value will be unity, in virtue of the above. Hence $xa = p$, as was to be deduced. We have thus shown the necessity of the polyhedral condition.

3. Ratios. In preparation for the sufficiency proof of the polyhedral condition, we shall consider a semigroup $\mathfrak{H}$ satisfying (C) and (P). It is our intention to introduce ratios, in more or less the same way as is usually done when $\mathfrak{H}$ is the set of natural numbers.

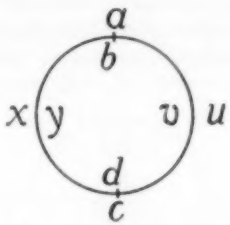

Fig. 3

Let a and b be any two elements of $\mathfrak{H}$. We shall designate by a/b the set of pairs of elements x,y such that $xa = yb$. If a/b is not the empty set we shall call it a *ratio*. Similarly we define I, (a), $(a)^{-1}$ as sets of pairs x,y such that $x = y$, $xa = y$, $x = ya$, respectively. With the help of the cancellation law, we can easily show that they are also ratios. In fact

$$(1) \qquad I = t/t, \qquad (a) = at/t, \qquad (a)^{-1} = t/at,$$

where t is an arbitrary element of $\mathfrak{H}$. We also note that

$$(2) \qquad (a) = (b) \text{ *if and only if* } a = b.$$

When can we say that two ratios are equal? We will prove:

(3) $a/b = c/d$ *if and only if there exist* x *and* y *belonging to* $\mathfrak{H}$ *such that* $xa = yb$ *and* $xc = yd$. (It is assumed here that a/b is in fact a ratio, and therefore not empty.)

The necessity of this condition follows directly from the definition of ratios. To prove its sufficiency, let us assume that the condition of the theorem holds, and also that $ua = vb$. Let us now apply the polyhedral condition to a simple polyhedron consisting of two edges, two faces, and two vertices (Fig. 3). The equations corresponding to three of the half-edges of this polyhedron are true by assumption, consequently the fourth must hold, viz. $uc = vd$. This argu-

ment works both ways, hence $ua = vb$ if and only if $uc = vd$, and therefore $a/b = c/d$, by definition.

We define multiplication of ratios as follows:

$$(a/d)(d/b) = a/b. \tag{4}$$

Thus two ratios may be multiplied to give another ratio, provided they can be written in the form a/d and d/b respectively such that a/b is a ratio. Is the product of the two ratios unique, if it exists? To answer this in the affirmative we must show:

(5) *If* $a/d = a'/d'$ *and* $d/b = d'/b'$, *then* $a/b = a'/b'$.

We may assume that $xa = yd$, $xa' = yd'$, $zd = wb$, $zd' = wb'$, $ua' = vb'$, and wish to prove that $ua = vb$. The result follows immediately if we consider the polyhedron consisting of two vertices, three faces, and three edges. (See Fig. 4.)

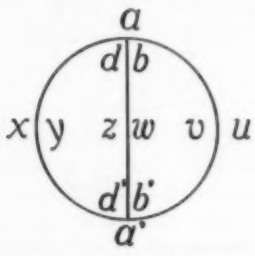

Fig. 4

I is the unit element under multiplication. For

$$(a/b)I = (a/b)(b/b) = a/b$$

by (1) and (4). The same applies to multiplication by I on the left. We note the existence of inverses; thus by (1) and (4),

$$(a/b)(b/a) = a/a = I.$$

In particular we see from (1) that $(a)^{-1}$ is the inverse of (a), as was anticipated by our notation. We find that

$$(a)(b) = (abt/bt)(bt/t) = abt/t = (ab).$$

Hence, in view of (2), the correspondence $a \to (a)$ maps $\mathfrak{H}$ isomorphically on a subset of the set of ratios.

We have embedded $\mathfrak{H}$ in the set of ratios. The latter has all properties of a group, except that it is not closed under multiplication, and associativity has not yet been shown to hold. We shall embed it in a larger set, in which multiplication is always defined and associative. It may be worth noting that, if $\mathfrak{H}$ is an Abelian semigroup, the ratios do form a group already.

4. Associativity. The set of ratios almost form a group, except that it is not closed under multiplication, so that the associative law, as usually stated, has no meaning. With some care however it is possible to enunciate an associative law even here. If we can bracket a sequence of ratios in such a way that they can be multiplied out to give a single ratio, then this "product" will be unique. To be more precise: we shall say that a finite sequence of ratios $(\ldots, a/b, b/c, \ldots)$ *contracts* into the sequence $(\ldots, a/c, \ldots)$. If a sequence of ratios reduces to a single ratio by iterated contraction, we shall call this ratio its *product*. The associative law then states:

(6) *If a sequence of ratios has a product, then this is unique.*

To prove (6), consider a sequence $S(0)$ of $n+1$ ratios. This contracts to $S(\pm 1)$, consisting of n ratios, which in turn contracts to $S(\pm 2)$, and so on, until we obtain a single ratio $S(\pm n)$. The choice of the plus sign refers to one method of iterated contraction, the minus to another. We must show that $S(+n) = S(-n)$.

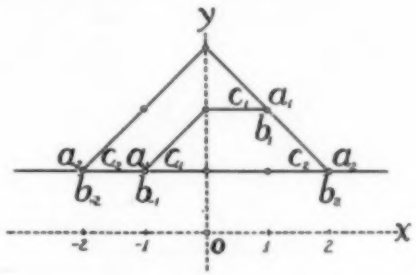

Fig. 5

If k is an integer between 0 and n, we write $i = \pm k$, and note that $S(i)$ has $n + 1 \mp k$ places or terms. Represent the jth place of $S(i)$ by the point (i,j) in the Cartesian plane. If $k \neq 0$, all but two terms of $S(\pm k \mp 1)$ reappear in $S(\pm k)$; join the corresponding points by straight lines. But two consecutive terms, say a_i/c_i and c_i/b_i are contracted into a_i/b_i. Join the two former points by straight lines to the latter, which will be called a *vertex*. Also join the two points (or vertices) $(\pm n, 1)$ to the point at infinity, along the line $y = 1$. A broken line joining two vertices, even if it passes through the point at infinity, will be called an *edge*. There are three edges meeting at every vertex. The simply connected regions into which the edges divide the plane will be called *faces*. Since the plane can be mapped on a sphere by an inverse stereographic projection, we obtain a concrete representation of an Eulerian polyhedron. We may also verify independently that $V = 2n$, $F = n + 2$, and $E = 3n$, so that $V + F - E = 2$. A simple case, for which $n = 2$, is illustrated by Fig. 5.

Consider the vertex corresponding to the contraction of

$$S(\pm k \mp 1) = (\ldots, a_i/c_i, c_i/b_i, \ldots) \text{ into } S(\pm k) = (\ldots, a_i/b_i, \ldots).$$

To the three angles formed at this vertex we assign a_i, c_i, and b_i, in this order, going from top to bottom, as shown in Fig. 5. By our construction, if a and b have been assigned to the upper, respectively lower angle at one end of any finite edge, and if this edge passes through any integral lattice point (i, j), then the jth term of $S(i)$ is a/b. But in the same way, we find that this term is c/d, where c and d correspond to the two angles at the other end of the edge. Thus, for any finite edge, we have a "proportion" $a/b = c/d$. We will prove that such a proportion also holds for the edge passing through the point at infinity.

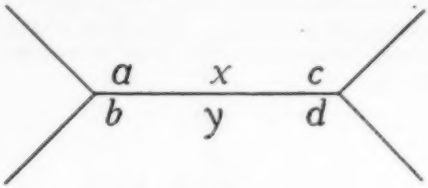

Fig. 6

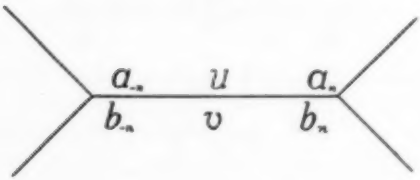

Fig. 7

In view of (3), the above proportion may be replaced by the two equations $xa = yb$ and $xc = yd$. Here x and y may be conveniently assigned to the two sides of the edge (see Fig. 6), and the two equations may be said to correspond to the two half-edges. Consider now the edge joining $(n, 1)$ and $(-n, 1)$ through the point at infinity. An appropriate transformation will bring the point at infinity into the finite part of the plane. Since a_{-n}/b_{-n} is a ratio, by definition, there exist elements u and v of $\mathfrak{H}$ such that $ua_{-n} = vb_{-n}$. We may assign u and v to the upper, respectively lower side of the edge depicted in Fig. 7, and the given equation will correspond to the left half of this edge. With the help of the polyhedral condition, we deduce the remaining equation $ua_n = vb_n$. It follows from (3) that $a_{-n}/b_{-n} = a_n/b_n$, i.e., $S(-n) = S(+n)$. This concludes the proof of the associative law.

5. Sufficiency of polyhedral condition. Two finite sequences of ratios, U and V, will be called *similar*, if there is a sequence W from which both can be obtained by repeated contraction. We will prove the following result:

(7) *If both U and V reduce to S by iterated contraction, then they are similar.*

First, suppose U contracts to S, so that $U = (P, a/c, c/b, Q)$ and $S = (P, a/b, Q)$, where P and Q may be empty sequences. Since V reduces to S, we may put $V = (X, Y, Z)$, where X, Y, and Z reduce to P, a/b, and Q respectively, by iterated contraction. It is easily seen that $W = (X, Y, b/c, c/b, Z)$ can be reduced to both U and V by repeated contraction, so that U and V are similar. Hence (7) holds when U reduces to S in one step.

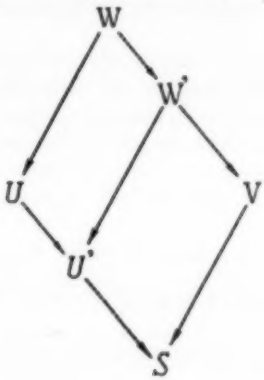

Fig. 8

Next, suppose U reduces to S in n steps, $n > 1$. Then U contracts to U' which reduces to S in $n - 1$ steps. By induction hypothesis, there exists a sequence W' which reduces to U' and V by iterated contraction. Since U reduces to U' in only one step, by the above, there exists a sequence W which reduces to both U and W' and therefore V. Hence U and V are similar, as was to be proved. (See Fig. 8 for an illustration of the second part of this proof.)

We are now in a position to show that similarity of sequences of ratios is an equivalence relation in the usual sense.

(8) *Similarity is symmetric, reflexive, and transitive.*

Its symmetry is obvious. Reflexivity follows from the fact that (S,I) contracts to S. To prove transitivity, let us assume that R is similar to S, and S is similar to T. Hence there exists a sequence U which reduces to both R and S, and a sequence V which reduces to both S and T (see Fig. 9). Since both U and V reduce to S, by (7) they can both be obtained from a sequence W

by repeated contraction. Now W reduces to R via U and to T via V, hence R is similar to T, as was to be proved.

In this connection we may also state:

(9) *If S is similar to S' and T is similar to T', then (S,T) is similar to (S',T').*

For, by repeated contraction, we obtain S and S' from U, T and T' from V, hence (S,T) and (S', T') from (U,V).

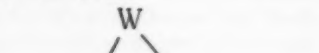

Fig. 9

A ratio may be regarded as a sequence of ratios with one term. When are two ratios similar?

(10) *Two ratios are similar if and only if they are equal.*

Because of reflexivity we know that equal ratios are similar. Conversely, let two ratios be similar. By definition, this means that both can be derived from the same sequence W by repeated contraction. From (6) we deduce that they are equal.

Let us denote by S^* the class of all sequences which are similar to S, so that $S^* = T^*$ if and only if S and T are similar, in view of (8). We define *multiplication* of similarity classes as follows:

$$S^*T^* = (S,T)^*. \tag{11}$$

From (9) we know that the product thus defined is unique.

Associativity becomes apparent if we write both $(S^*T^*)U^*$ and $S^*(T^*U^*)$ as $(S,T,U)^*$. The unit element under multiplication is I^*, since both (S,I) and (I,S) are similar to S; for both are obtained by contracting (I,S,I). If T contains the reciprocals of the ratios of S in reverse order, then both (S,T) and (T,S) reduce to and are therefore similar to I; thus T^* may be regarded as the inverse of S^* under multiplication. We have thus proved that the similarity classes form a group $\mathfrak{G}$, with multiplication defined by (11).

The correspondence $a/b \to (a/b)^*$ is a homomorphic mapping of the set of ratios on a subset of $\mathfrak{G}$. For, by (11) and (4),

$$(a/b)^*(b/c)^* = (a/b,b/c)^* = (a/c)^* = ((a/b)(b/c))^*.$$

More than this, the mapping is isomorphic. For if $(a/b)^* = (c/d)^*$ then a/b and c/d are similar, hence $a/b = c/d$, by (10). The correspondence $a/b \rightarrow (a/b)^*$ therefore embeds the set of ratios in $\mathfrak{G}$. But the correspondence $a \rightarrow (a)$ embeds the semigroup $\mathfrak{H}$ in the set of ratios, as we have shown in §3. Hence the correspondence $a \rightarrow (a)^*$ embeds $\mathfrak{H}$ in $\mathfrak{G}$. This establishes the sufficiency of the polyhedral condition.

6. Application to Abelian semigroups. Let $\mathfrak{H}$ be an Abelian semigroup with cancellation law. Although it is not difficult to show directly that $\mathfrak{H}$ is immersible in a group (namely the set of ratios), we shall test the usefulness of the polyhedral condition, by showing independently that the latter holds in $\mathfrak{H}$.

Let elements of $\mathfrak{H}$ be assigned to all angles and sides of any given polyhedron. Of the equations corresponding to the half-edges we will assume that all but one hold, and we wish to deduce the remaining equation. As in the necessity proof of the polyhedral condition, we introduce a triangulation and replace each equation $xa = yb$ by two equations $xa = p$ and $yb = p$ corresponding to triangles. We may assume then that all but two of these latter equations are given.

If we reverse the direction assigned to all half-edges, we obtain a cyclic orientation for each triangle, enabling us to distinguish clockwise and counter-clockwise triangles. The triangles are thus divided into two classes, so that triangles with a common edge do not belong to the same class. We will write the equation corresponding to a triangle of the first class as $xa = p$, and the equation corresponding to a triangle of the second class as $p = yb$, making a careful distinction between the two sides of each equation. Now multiply all $4E - 2$ given equations together, after their sides have been thus arranged. It will be observed that the four variables belonging to the half-edge whose equation is to be deduced occur once in the product equation. All other variables occur twice, once on each side of the product equation, and may therefore be cancelled, by (C). There results an equation containing four variables, and it is easily seen that this is in fact the equation we wish to deduce. Hence the polyhedral condition is satisfied, as was to be proved.

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ON EXPONENTIAL SUMS OVER AN ALGEBRAIC NUMBER FIELD

LOO-KENG HUA

1. Introduction

LET K be an algebraic field of degree n over the rational field, and let $\mathfrak{d}$ be the ground ideal (differente) of the field. Let

$$f(x) = \mathfrak{a}_k x^k + \ldots + \mathfrak{a}_1 x + \mathfrak{a}_0$$

be a polynomial of the kth degree with coefficients in the field K, and let $\mathfrak{a}$ be the fractional ideal generated by $\mathfrak{a}_k, \ldots, \mathfrak{a}_1$, that is, $\mathfrak{a} = (\mathfrak{a}_k, \ldots, \mathfrak{a}_1)$. Suppose $\mathfrak{a}\mathfrak{d} = \mathfrak{r}/\mathfrak{q}$, where $\mathfrak{r}$ and $\mathfrak{q}$ are two relatively prime integral ideals, and

$$S(f(x), \mathfrak{q}) = S(f(x)) = S(\mathfrak{q}) = \sum_{x \bmod \mathfrak{q}} e^{2\pi i \operatorname{tr}(f(x))},$$

where x runs over a complete residue system, mod $\mathfrak{q}$. It is the aim of the paper to prove the following:

THEOREM 1. *For any given $\epsilon > 0$, we have*

$$S(f(x), \mathfrak{q}) = O(N(\mathfrak{q})^{1-1/k+\epsilon})$$

where the constant implied by the symbol O depends only on k, n and ϵ.

As usual, we use $\operatorname{tr}(\mathfrak{a})$and $N(\mathfrak{q})$ to denote the trace of a number $\mathfrak{a}$ and the norm of an ideal $\mathfrak{q}$ of K respectively.

This is a generalization of a theorem of the author's [1] over the rational field. The method used here is simpler and quite different from the original one.

1. A theorem on congruences

THEOREM 2. *Let $\mathfrak{p}$ be a prime ideal and let $s(x)$ be a polynomial with integral coefficients,* mod $\mathfrak{p}$. *Let $\mathfrak{a}$ be a root of multiplicity m of the congruence*

$$s(x) \equiv 0 \pmod{\mathfrak{p}}.$$

Let λ be an integer, divisible by $\mathfrak{p}$ but not by $\mathfrak{p}^2$, and let u be the greatest integer such that $\mathfrak{p}^u$ divides all the coefficients of $s(\lambda x + \mathfrak{a}) - s(\mathfrak{a})$. Let

$$t(x) \equiv \lambda^{-u}(s(\lambda x + \mathfrak{a}) - s(\mathfrak{a})) \pmod{\mathfrak{p}}$$

be a polynomial with integral coefficients. Then $u \leqslant m$, and the congruence

$$t(x) \equiv 0 \pmod{\mathfrak{p}}$$

has at most m solutions.

Received September 14, 1949.

Proof. Without loss of generality, we may assume that $\mathfrak{a} = 0$. Then

$$s(x) = x^m s_1(x) + s_2(x),\ s_1(0) \not\equiv 0 \pmod{\mathfrak{p}}$$

where $s_2(x)$ is a polynomial of degree less than m and all its coefficients are divisible by $\mathfrak{p}$. Now we have

$$s(\lambda x) = \lambda^m x^m s_1(\lambda x) + s_2(\lambda x).$$

Since the coefficient of x^m is equal to $\lambda^m s_1(0)$ which is not divisible by $\mathfrak{p}^{u+1}$, we have $u \leqslant m$.

Since $\lambda^{-u} s(\lambda x)$ is congruent to a polynomial of degree not exceeding m, mod $\mathfrak{p}$, the theorem follows.

Remark. u is independent of the choice of λ. In fact, let λ' be another integer having the same property, then we have an integer τ such that

$$\lambda \equiv \lambda' \tau \pmod{\mathfrak{p}^{u+1}},\ \mathfrak{p} \nmid \tau.$$

Then

$$s(\lambda x + \mathfrak{a}) - s(\mathfrak{a}) \equiv s(\lambda'(\tau x) + \mathfrak{a}) - s(\mathfrak{a}) \pmod{\mathfrak{p}^{u+1}}$$

3. Several lemmas concerning algebraic numbers

Let $\mathfrak{g}$ be an ideal, fractional or integral, and $\mathfrak{a}$ be an integral ideal. It is clear that $\mathfrak{g} \mid \mathfrak{g}\mathfrak{a}$.

Now we divide the elements of $\mathfrak{g}$ into residue classes according to the modulus $\mathfrak{g}\mathfrak{a}$. The number of different classes is known to be $N(\mathfrak{a})$. We take an element from each class; the set so formed is called a complete residue system of $\mathfrak{g}$, mod $\mathfrak{g}\mathfrak{a}$.

The definition of the ground ideal $\mathfrak{d}$ can be stated in the following way:

$\mathfrak{d}^{-1}$ is the aggregate of all numbers ξ of K

such that

$$e^{2\pi i \operatorname{tr}(\xi\mathfrak{a})} = 1$$

for all integers $\mathfrak{a}$ of K. Consequently, if β belongs to $(\mathfrak{q}\mathfrak{d})^{-1}$ and $\mathfrak{a}_1 \equiv \mathfrak{a}_2 \pmod{\mathfrak{q}}$, then

$$e^{2\pi i \operatorname{tr}(\beta\mathfrak{a}_1)} = e^{2\pi i \operatorname{tr}(\beta\mathfrak{a}_2)}.$$

This asserts that the sum $S(f(x), \mathfrak{q})$, which was defined at the beginning of the paper, is independent of the choice of the residue system, mod $\mathfrak{q}$.

THEOREM 3. *Let $\mathfrak{q}$ be an integral ideal. As ξ runs over a complete residue system of* $(\mathfrak{q}\mathfrak{d})^{-1}$, mod $\mathfrak{d}^{-1}$, *we have, for integral* $\mathfrak{a}$,

$$\sum_{\xi} e^{2\pi i \operatorname{tr}(\xi\mathfrak{a})} = \begin{cases} N(\mathfrak{q}) & \text{if } \mathfrak{q} \mid \mathfrak{a}, \\ 0 & \text{if } \mathfrak{q} \nmid \mathfrak{a}. \end{cases}$$

Proof. If $\mathfrak{q} \mid \mathfrak{a}$, then $\xi\mathfrak{a}$ belongs to $\mathfrak{d}^{-1}$. Then $e^{2\pi i \operatorname{tr}(\xi\mathfrak{a})} = 1$ for all ξ. Hence, we have the first conclusion.

If $\mathfrak{q} \nmid \mathfrak{a}$, there is an element ξ_0, which belongs to $(\mathfrak{bq})^{-1}$, but $\xi_0\mathfrak{a}$ does not belong to $\mathfrak{b}^{-1}$. In fact, if for all ξ_0 belonging to $(\mathfrak{bq})^{-1}$ we have $\xi_0\mathfrak{a}$ belonging to $\mathfrak{b}^{-1}$, then we have

$$\mathfrak{b}^{-1} \mid \mathfrak{a}(\mathfrak{bq})^{-1}.$$

Consequently $\mathfrak{q} \mid \mathfrak{a}$. This is impossible. By the definition of $\mathfrak{b}^{-1}$ there is an integer γ such that

$$e^{2\pi i \operatorname{tr}(\gamma\xi_0\mathfrak{a})} \not\equiv 1.$$

Since $\gamma\xi_0$ belongs to $(\mathfrak{bq})^{-1}$, we can write $\gamma\xi_0 = \xi_1$. Then

$$\begin{aligned}\sum_{\xi} e^{2\pi i \operatorname{tr}(\xi\mathfrak{a})} &= \sum_{\xi} e^{2\pi i \operatorname{tr}((\xi+\xi_1)\mathfrak{a})} \\ &= e^{2\pi i \operatorname{tr}(\xi_1\mathfrak{a})} \cdot \sum_{\xi} e^{2\pi i \operatorname{tr}(\xi\mathfrak{a})}.\end{aligned}$$

Thus we have the second conclusion of our theorem.

4. Proof of the theorem for $\mathfrak{q} = \mathfrak{p}$

In case $\mathfrak{q}$ is a prime ideal $\mathfrak{p}$, the exponential sum considered here reduces to a type of exponential sum over a finite field which has been discussed before [2]. But the author could not find an easy way to establish an explicit relationship between the exponential sums considered here and those over a finite field. Also, for the sake of completeness, the following proof is included here. The method is an adaptation of one due to Mordell [3].

THEOREM 4. *We have*

$$|S(f(x), \mathfrak{p})| \leqslant k^n N(\mathfrak{p})^{1-1/k}.$$

Proof. Without loss of generality, we may assume that $\mathfrak{a}_k$ does not belong to $\mathfrak{b}^{-1}$, for otherwise

$$S(f(x), \mathfrak{p}) = S(f(x) - \mathfrak{a}_k x^k, \mathfrak{p}),$$

since $e^{2\pi i \operatorname{tr}(\mathfrak{a}_k x^k)} = 1$ for all integral x. Thus we now assume that $\mathfrak{a}_k$ belongs to $(\mathfrak{pb})^{-1}$ but not to $\mathfrak{b}^{-1}$. The theorem is trivial for $N(\mathfrak{p}) \leqslant k^n$, since

$$|S(f(x), \mathfrak{p})| \leqslant N(\mathfrak{p}) \leqslant k^n N(\mathfrak{p})^{1-1/k}.$$

Now we assume $N(\mathfrak{p}) > k^n$ and consequently $\mathfrak{p} \nmid k$! We have

$$|S(f(x))|^{2k} = \frac{1}{N(\mathfrak{p})(N(\mathfrak{p}) - 1)} \sum_{\lambda \bmod \mathfrak{p}}{}' \sum_{\mu \bmod \mathfrak{p}} |S(f(\lambda x + \mu))|^{2k},$$

where λ runs over a reduced residue system, mod $\mathfrak{p}$. Write

$$f(\lambda x + \mu) = \beta_k x^k + \ldots + \beta_0,$$

where

$$(1) \qquad \beta_k \equiv \mathfrak{a}_k \lambda^k \pmod{\mathfrak{b}^{-1}},$$

$$(2) \qquad \beta_{k-1} \equiv k\mathfrak{a}_k \lambda^{k-1} + \mathfrak{a}_{k-1}\lambda^{k-1} \pmod{\mathfrak{b}^{-1}},$$

and so on.

For fixed β_k, β_{k-1}, ... belonging to $(\mathfrak{p}\mathfrak{d})^{-1}$, the number of integers λ and μ does not exceed k. In fact, (1) asserts that $\beta_k - a_k\lambda^k$ belongs to $\mathfrak{d}^{-1}$. (β_k and a_k belong to $(\mathfrak{p}\mathfrak{d})^{-1}$.) There is an integer τ belonging to $\mathfrak{p}\mathfrak{d}$ but not to $\mathfrak{p}$. Consequently τa_k and $\tau\beta_k$ are integers and $\mathfrak{p} \nmid \tau a_k$; the congruence $\tau\beta_k \equiv \tau a_k\lambda^k \pmod{\mathfrak{p}}$ has evidently at most k solutions. For a fixed λ, the same argument proves that μ is uniquely determined by (2), since $\mathfrak{p} \nmid k$.

Therefore, we have

$$\left| S(f(x), \mathfrak{p}) \right|^{2k} \leqslant \frac{k}{N(\mathfrak{p})\ (N(\mathfrak{p})-1)} \sum_{\beta_k} \cdots \sum_{\beta_1} \left| S(\beta_k x^k + \ldots + \beta_1 x) \right|^{2k},$$

where each β runs over a complete residue system of $(\mathfrak{p}\mathfrak{d})^{-1}$, mod $\mathfrak{d}^{-1}$.

We have

$$\sum_{\beta_k} \cdots \sum_{\beta_1} \left| S(\beta_k x^k + \ldots + \beta_1 x) \right|^{2k} = \sum_{\beta_k} \cdots \sum_{\beta_1} \sum_{x_1} \cdots \sum_{x_k} \sum_{y_1} \cdots \sum_{y_k} e^{2\pi i \operatorname{tr}(\psi)}$$
$$= N(\mathfrak{p})^k M,$$

where

$$\psi = \beta_k(x_1^k + \ldots + x_k^k - y_1^k - \ldots - y_k^k) + \beta_{k-1}(x_1^{k-1} + \ldots + x_k^{k-1} - y_1^{k-1} - \ldots - y_k^{k-1}) + \ldots + \beta_1(x_1 + \ldots + x_k - y_1 - \ldots - y_k),$$

and, by Theorem 3, M is equal to the number of solutions of the system of congruences

$$x_1^h + \ldots + x_k^h \equiv y_1^h + \ldots + y_k^h, \text{ mod } \mathfrak{p}, \qquad 1 \leqslant h \leqslant k.$$

By a theorem on symmetric functions, we deduce immediately

$$(X - x_1) \ldots (X - x_k) \equiv (X - y_1) \ldots (X - y_k), \text{ mod } \mathfrak{p},$$

since $\mathfrak{p} \nmid k!$ Then we have that $x_1, \ldots, x_k$ are a permutation of $y_1, \ldots, y_k$ and then

$$M \leqslant k!N(\mathfrak{p})^k.$$

Consequently, we have

$$\left| S(f(x), \mathfrak{p}) \right|^{2k} \leqslant \frac{k \cdot k!}{N(\mathfrak{p})(N(\mathfrak{p})-1)} N(\mathfrak{p})^{2k}$$
$$\leqslant 2k \cdot k!N(\mathfrak{p})^{2k-2}$$
$$\leqslant k^{2k}N(\mathfrak{p})^{2k-2}$$

and the theorem follows.

5. Proof of the theorem for $\mathfrak{q} = \mathfrak{p}^l$

THEOREM 5. *If $\mathfrak{q} = \mathfrak{p}^l$, and $\mathfrak{p}$ is a prime ideal, then*

$$|S(f(x), \mathfrak{p}^l)| \leqslant k^{2n+1}N(\mathfrak{p}^l)^{1-1/k}. \tag{1}$$

Proof. Let

$$\mathfrak{b} = (ka_k, (k-1)a_{k-1}, \ldots, 2a_2, a_1).$$

Evidently $\mathfrak{a} \mid \mathfrak{b}$. Let t be the highest exponent of $\mathfrak{p}$ dividing $\mathfrak{b}\mathfrak{a}^{-1}$. Let m be the number of solutions, multiplicities being counted, of the congruence

$$f'(x) \equiv 0 \pmod{\mathfrak{p}^{t+1-l}} \tag{2}$$

as x runs over a complete residue system, mod $\mathfrak{p}$. (We have $m \leqslant k-1$.)

Evidently, (1) is a conseqeunce of the sharper result

$$|S(f(x), \mathfrak{p}^l)| \leqslant k^{2n} \max(1, m)\, N(\mathfrak{p}^l)^{1-1/k}. \tag{3}$$

If $t \geqslant 1$, then $\mathfrak{p}^t$ divides at least one of the integers $k, k-1, \ldots, 1$. Then

$$N(\mathfrak{p}^t) \leqslant k^n,$$

that is

$$N(\mathfrak{p}) \leqslant k^{n/t}. \tag{4}$$

Suppose that $l < 2(t+1)$. For $t = 0$, we have $l = 1$ and (3) follows from Theorem 4. If $t \geqslant 1$, then, by (4)

$$\begin{aligned} |S(f(x), \mathfrak{p}^l)| &\leqslant N(\mathfrak{p})^l \leqslant (N(\mathfrak{p}))^{l(1-1/k)}\ (N(\mathfrak{p}))^{(2t+1)/k} \\ &\leqslant N(\mathfrak{p})^{l(1-1/k)}\ k^{n(2+1/t)/k} \\ &\leqslant k^{2n} \cdot N(\mathfrak{p})^{l(1-1/k)}. \end{aligned}$$

Therefore (3) is true for $l \leqslant 2t+1$. Now we assume that $l \geqslant 2(t+1)$ and that (3) is true for smaller l.

Let $\mu_1, \ldots, \mu_r$ be the distinct roots of (2) with multiplicities $m_1, \ldots, m_r$ respectively. Then $m_1 + \ldots + m_r = m$. Evidently

$$S(f(x)) = \sum_x e^{2\pi i \operatorname{tr}(f(x))} = \sum_\nu \sum_{x \equiv \nu \pmod{\mathfrak{p}}} e^{2\pi i \operatorname{tr}(f(x))} = \sum_\nu S_\nu$$

say, where ν runs over a complete residue system, mod $\mathfrak{p}$. If ν is not one of the μ's then, letting

$$x = y + \lambda^{l-t-1} z,$$

where λ is an integer belonging to $\mathfrak{p}$ but not to $\mathfrak{p}^2$, we have

$$\begin{aligned} S_\nu &= \sum_{\substack{y \bmod \mathfrak{p}^{l-t-1} \\ y \equiv \nu \pmod{\mathfrak{p}}}} \sum_{z \bmod \mathfrak{p}^{t+1}} e^{2\pi i \operatorname{tr}(f(y) + \lambda^{l-t-1} z f'(y))} \\ &= \sum e^{2\pi i \operatorname{tr}(f(y))} \sum_{z \bmod \mathfrak{p}^{t+1}} e^{2\pi i \operatorname{tr}(\lambda^{l-t-1} z f'(y))} \\ &= 0 \end{aligned}$$

by Theorem 3, since $\mathfrak{p}^{t+1-l} \nmid f'(y)$.

Therefore

$$|S(f(x))| \leqslant \sum_{s=1}^{r} \left| \sum_{x \bmod \mathfrak{p}^l - 1} e^{2\pi i \operatorname{tr}(f(\mu_s + \lambda y))} \right|$$

$$= \sum_{s=1}^{r} \left| \sum_{x \bmod \mathfrak{p}^l - 1} e^{2\pi i \operatorname{tr}(f(\mu_s + \lambda y) - f(\mu_s))} \right|$$

$$= \sum_{s=1}^{r} N(\mathfrak{p})^{\sigma_s - 1} S(f(\mu_s + \lambda y) - f(\mu_s), \mathfrak{p}^{l-\sigma_s}), \tag{5}$$

where σ_s is defined in the following way: Let $\mathfrak{c}$ be the ideal generated by the coefficients of

$$f_s(y) = f(\mu_s + y) - f(\mu_s).$$

Evidently $\mathfrak{a}$ divides $\mathfrak{c}$, and σ_s is the highest power of $\mathfrak{p}$ dividing $\mathfrak{c}\mathfrak{a}^{-1}$. Also, if $l \leqslant \sigma_s$, we use the conventional meaning

$$S(f(\mu_s + y) - f(\mu_s), \mathfrak{p}^{l-\sigma_s}) = \mathfrak{p}^{l-\sigma_s}.$$

Now we are going to prove that

$$1 \leqslant \sigma_s \leqslant k. \tag{6}$$

If (6) is not true, then $\mathfrak{p}^{-l+k+1}$ divides all the coefficients of $f(\sigma_s + y) - f(\sigma_s)$; that is

$$\mathfrak{p}^{-l+k+1} \left| \frac{f^{(r)}(\mu_s)}{r!} \lambda^r, \right. \qquad 1 \leqslant r \leqslant k.$$

Consequently

$$\mathfrak{p}^{-l+1} \left| \frac{f^{(r)}(\mu_s)}{r!}, \right.$$

which is equal to a_r plus a linear combination of $a_k, \ldots, a_{r-1}$ with integral coefficients. Thus we deduce successively $\mathfrak{p}^{-l+1} | a_k$, $\mathfrak{p}^{-l+1} | a_{k-1}, \ldots, \mathfrak{p}^{-l+1} | a_1$. This contradicts $\mathfrak{q} = \mathfrak{p}^l$.

From (5) and (6), we have, for $l \geqslant \max(\sigma_1, \ldots, \sigma_r)$,

$$|S(f(x), \mathfrak{p}^l)| \leqslant \sum_{s=1}^{r} N(\mathfrak{p})^{\sigma_s(1-1/k)} |S(f_s(y), \mathfrak{p}^{l-\sigma_s})|.$$

By the hypothesis of induction, we have

$$|S(f(x), \mathfrak{p}^l)| \leqslant k^{2n} \sum_{s=1}^{r} N(\mathfrak{p})^{\sigma_s(1-1/k)} m_s N(\mathfrak{p})^{(l-\sigma_s)(1-1/k)}$$

$$= k^{2n} m N(\mathfrak{p})^{l(1-1/k)}.$$

In case $l \leqslant \max(\sigma_1, \ldots, \sigma_r)$, we have $l \leqslant k$ and, by (5)

$$|S(f(x))| \leqslant r \mathfrak{p}^{l-1} \leqslant m \mathfrak{p}^{l(1-1/k)}.$$

We have (3) and consequently (1). (Notice that if $\sum_{s=1}^{r} m_s = 0$, the method shows that $S(f(x)) = 0$, if $l \geqslant 2(t+1)$.)

THEOREM 6. *If* $(q_1, q_2) = 1$ *and* $f(0) = 0$, *then there are polynomials* $f_1(x)$ *and* $f_2(x)$ *each of degree* k *such that*

$$S(f(x), q_1q_2) = S(f_1(x), q_1)\ S(f_2(x), q_2).$$

Proof. We can find two integers λ_1 and λ_2 such that

$$(\lambda_1, q_1q_2) = q_2,\ (\lambda_2, q_1q_2) = q_1.$$

Putting

$$x = \lambda_1 y_2 + \lambda_2 y_1,$$

then, as y_1 and y_2 run over complete residue systems mod q_1 and mod q_2 respectively, x runs over a complete residue system, mod q_1q_2. Then

$$\begin{aligned} S(f(x), q_1q_2) &= \sum_{y_1 \bmod q_1} \sum_{y_2 \bmod q_2} e^{2\pi i \operatorname{tr}(f(\lambda_1 y_2 + \lambda_2 y_1))} \\ &= \sum_{y_1 \bmod q_1} e^{2\pi i \operatorname{tr}(f(\lambda_2 y_1))} \sum_{y_2 \bmod q_2} e^{2\pi i \operatorname{tr}(f(\lambda_1 y_2))} \\ &= S(f_1(x), q_1))\ \ S(f_2(x), q_2)\ , \end{aligned}$$

where $f_1(x) = f(\lambda_2 x)$ and $f_2(x) = f(\lambda_1 x)$. Now we have to verify that the ideal generated by the coefficients of $f_1(x)$ can be expressed as $\mathfrak{r}(\mathfrak{b}\mathfrak{q}_1)^{-1}$, where $\mathfrak{r}$, $\mathfrak{q}$ are relatively prime integral ideals, but this is quite evident.

6. Proof of Theorem 1

Let $$\mathfrak{q} = \mathfrak{p}_1^{l_1} \ldots \mathfrak{p}_s^{l_s}.$$

Then we have, by repeated application of Theorem 6,

$$S(f(x), \mathfrak{q}) = \prod_{i=1}^{s} S(f_i(x), \mathfrak{p}_i^{l_i})\ .$$

By Theorem 5, we have

$$\begin{aligned} |\ S(f(x), \mathfrak{q})\ | &\leqslant \sum_{i=1}^{s} k^{2n+1}\ N(\mathfrak{p}_i^{l_i})^{1-1/k} \\ &\leqslant \sum_{i=1}^{s} (1+l_i)^{(2n+1)\log k/\log 2}\ N(\mathfrak{p}_i^{l_i})^{(1-1/k)} \\ &= d(\mathfrak{q})^{(2n+1)\log k/\log 2}\ N(\mathfrak{q})^{1-1/k} \\ &= O(N(\mathfrak{q})^{1-1/k+\epsilon}) \end{aligned}$$

where $d(\mathfrak{q})$ denotes the number of divisors of $\mathfrak{q}$.

Remarks. The previous method is practically an algorithm; more precisely, for a given polynomial, if we know the value of $S(f(x), \mathfrak{p}^l)$, $l \leqslant 2t + 1$, then we can find the value of $S(f(x), \mathfrak{p}^l)$.

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A PROPERTY OF FAREY SEQUENCES, WITH APPLICATIONS TO qTH POWER RESIDUES

PASQUALE PORCELLI AND GORDON PALL

THE Farey sequence of order $h-1$ consists of the reduced rational fractions from 0 to 1 inclusive, with denominators less than h, and arranged in order of magnitude. Thus, if $h=6$, the sequence is

(1) $$0/1,\ 1/5,\ 1/4,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 3/4,\ 4/5,\ 1/1.$$

It is well known that for any two consecutive terms r/s and t/u,

(2) $$ts - ru = 1, \quad s + u \geqslant h.$$

The principal result of this note is the observation that, by means of a Farey sequence, there can be written a complete system of residues, modulo an integer n, this system being expressed by fractions of the form a/u, with a and u suitably bounded.

THEOREM 1. *The integers of the sequence* $1, 2, \ldots, n$ *are obtained in order, each integer exactly once, in the sequence of sequences associated in the following manner with the terms* t/u *of a Farey sequence. With the term* t/u *is associated the sequence (possibly, for small* n*, empty) of positive integers*

(3) $$(nt + a)/u, \quad -n/(s+u) < a \leqslant n/(u+w),$$

where r/s *and* v/w *denote, respectively, the predecessor and successor of* t/u *in the Farey sequence (non-positive or positive values* a *being omitted if* t/u *is* $0/1$ *or* $1/1$*, respectively), and* a *runs over these integral values in the stated interval such that* $(nt + a)/u$ *is an integer.*

To illustrate the theorem and its later application, the sequences in the case $h=6$ associated with the terms in (1) are

$$a\ (0 \leqslant a \leqslant n/6);\ (n+a)/5 \quad (-n/6 < a \leqslant n/9,\ a \equiv -n \bmod 5);$$
$$(n+a)/4 \quad (-n/9 < a \leqslant n/7,\ a \equiv -n \bmod 4); \ldots;$$
$$(2n+a)/5 \quad (-n/8 < a \leqslant n/7,\ a \equiv -2n \bmod 5); \ldots.$$

It will be noted that the sequences associated with different terms of the Farey sequence do not overlap, and between them exactly cover the interval 1 to n. Also, $|a|$ does not exceed $n/6$, and if n is prime to all the denominators in (1), then the expression a/u (where $1 \leqslant u \leqslant 5$ and $|a| \leqslant n/6$) gives every residue mod n, with possibly some overlapping.

The proof of the theorem depends on the simple observation that, if r/s and t/u are consecutive terms of the Farey sequence, then

(4) $$\frac{rn + \dfrac{n}{s+u}}{s} = \frac{tn - \dfrac{n}{s+u}}{u},$$

this reducing to (2_1). It follows that the *real* numbers from 1 to n are covered by allowing a to assume all real values in the successive intervals in (3). The integers in this interval are therefore obtained by expressing the condition on a for $(nt + a)/u$ to be an integer: namely, a must be an integer congruent to $-nt \bmod u$.

If n is an odd prime p, and $p > 2h - 2$, then since $u + w < p$, equality cannot hold in the last part of (3). Thus, the sequence $1, 2, \ldots, p - 1$ is obtained by allowing a to range over the integers not exceeding numerically the greatest integers in $n/(s + u)$ and $n/(u + w)$. Since $s + u \geqslant h$ and $u + w \geqslant h$, we have the following result.

THEOREM 2. *Let p be an odd prime, h be a positive integer, $p > 2h - 2$, $k = [p/h]$. The sequence $1, \ldots, p - 1$ is obtained, possibly with some overlapping, by giving a the positive integer values from 1 to k inclusive such that $(pt \pm a)/u$ are integers. Negative and positive values $\pm a$ are omitted if t/u is $0/1$ or $1/1$ respectively.*

Since $1 \leqslant u < h$ and $1 \leqslant a \leqslant k$, the following is an immediate corollary.

THEOREM 3. *Let p be an odd prime, h and q be positive integers, $p > 2h - 2$, $k = [p/h]$, and D denote the residue modulo p of the qth power of some integer prime to p. Then one of the numbers Du^q $(u = 1, 2, \ldots, h - 1)$ is congruent to at least one of the numbers $(\pm 1)^q, (\pm 2)^q, \ldots, (\pm k)^q$ modulo p.*

It is interesting to notice that if $q = 2$ and $h = 2$, this reduces to the familiar proposition that the squares of $1, 2, \ldots, \frac{1}{2}(p - 1)$ constitute a complete system of quadratic residues mod p.

Theorem 3 permits a considerable reduction in the work of solving the congruence $x^q \equiv D \pmod p$, especially when p is beyond the range of existing tables of indices. The single congruence can be replaced by a system in which D is replaced in turn by Du^q $(u = 1, 2, \ldots, h - 1)$ reduced mod p. If D' denotes any one of these residues, the values $D' + yp$ need to be constructed only up to the limits $(\pm k)^q$, where $k = [p/h]$. The possible values y can be restricted by the method of exclusion. Further restrictions on y can be obtained from the property that the quantities (3) are integral, and by examining the Farey series for any given h, the limit k can be replaced by possibly smaller limits $p/(s + u)$ or $p/(u + w)$ for each particular value of u. In the case $q = 2$, by taking h approximately equal to $p^{\frac{1}{3}}$, the amount of work is reduced by a factor of the order of size of $\frac{1}{2}p^{\frac{1}{3}}$, and (as may be more important) the effective range of a table of squares (or qth powers) is greatly increased. Thus, by taking $h = [p^{\frac{1}{3}}]$, primes up to 10^8 can be handled with a table of squares up to 10000^2. Note, finally that the modulus need not be assumed to be a prime.

Thanks are due to the referee who pointed out an error in our earlier, different proof of Theorem 1, in which we overlooked the possibility that for some n's, the sequence (3) may be vacuous.

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A COMPACTNESS THEOREM FOR AFFINE EQUIVALENCE-CLASSES OF CONVEX REGIONS

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In some parts of the Geometry of Numbers it is convenient to know that certain affine invariants associated with convex regions attain their upper and lower bounds. A classical example is the quotient of the critical determinant by the content (if the region is symmetrical) for which Minkowski determined the exact lower bound 2^{-n}. The object of this paper is to prove that for continuous functions of bounded regions the bounds are attained. The result is, of course, deduced from the selection theorem of Blaschke, and itself is a compactness theorem about the space of affine equivalence-classes.

1. Introduction. We consider affine transformations σ of Euclidean n-space (E^n) given by

$$(\sigma x)_i = y_i = \sum_{j=1}^{n} \sigma_{ij}x_j + \sigma_{i0} \qquad (i = 1, \ldots, n).$$

With each σ is associated the matrix, or homogeneous affine transformation

$$y_i = \sum_{j=1}^{n} \sigma_{ij}x_j \qquad (i = 1, \ldots, n).$$

This matrix will be denoted by $\mu(\sigma)$. The determinant of the matrix $\mu(\sigma)$ is called the determinant of σ, written $\det \sigma$. The mappings $\sigma \to \mu(\sigma) \to \det \sigma$ are homomorphic and $|\det \sigma|$ represents the factor by which the transformation alters content.

The affine transformations for which $\det \sigma \neq 0$ form a group, which will be denoted by G.

We define a *convex body* to be a bounded closed convex set *with inner points* in E^n. If $\sigma \in G$, σK is defined by the relation

$$x \in K \leftrightarrow \sigma x \in \sigma K.$$

Let $f(K)$ be a function defined on the space $\mathfrak{C}$ of all convex bodies in E^n: f is called an *affine invariant* if $f(K) = f(\sigma K)$ for all $\sigma \in G$, $K \in \mathfrak{C}$.

$\mathfrak{C}$ can be regarded as a metric space, for if $\delta(K, K')$ is the greatest distance of a point of K from K', i.e.,

$$\delta(K, K') = \sup_{x \in K} \left(\inf_{y \in K'} |x - y| \right),$$

then

$$\eta(K, K') = \max \left(\delta(K, K'), \delta(K', K)\right)$$

is a metric [1, p. 34].

Received May 16, 1949.

Define an *affine class* to be a set consisting of all convex bodies that can be derived from a fixed one by means of transformations of G. Let $\mathfrak{C}^*$ be the set of all affine classes. The topology of $\mathfrak{C}$ induces a topology on $\mathfrak{C}^*$, if we define a set of classes to be open in $\mathfrak{C}^*$ if and only if the union of these classes is open in $\mathfrak{C}$. Our main result is

THEOREM 1. *$\mathfrak{C}^*$ is a compact metric space.*

A weaker form of the result, often useful in applications, is

COROLLARY. *If $f(K)$ is an affine-invariant continuous real-valued function on $\mathfrak{C}$, then the upper and lower bounds of f are attained.*

2. Properties of G. A transformation $\sigma \in G$ is said to be *orthogonal* if the matrix $\mu(\sigma)$ is an orthogonal matrix and if also the terms σ_{i0} all vanish. The orthogonal transformations form a group O.

Now it is well known that every non-singular matrix can be expressed as the product of a positive definite symmetric matrix and an orthogonal one; but then the symmetric matrix can, by the usual reduction of quadratic forms, be written UDU^{-1}, where U is orthogonal and D is a diagonal matrix. It follows that every matrix can be written in the form

$$U_1 D U_2,$$

where the U are both orthogonal matrices and D is a diagonal one.

Applying the homomorphism $\sigma \to \mu(\sigma)$, we find that, if $\sigma \in G$ then σ can be put in the form

$$\sigma = \alpha_1 \delta \alpha_2, \tag{1}$$

where $\alpha_1, \alpha_2 \in O$ and δ is such that $\mu(\delta)$ is a diagonal matrix, i.e.,

$$\delta_{ij} = 0 \qquad (j \neq 0, i).$$

With each element σ of G associate the point of $E^{n(n+1)}$ whose coordinates are the $n(n+1)$ numbers

$$\sigma_{ij} \qquad (i = 1, \ldots, n;\ j = 0, \ldots, n).$$

The elements of G are thus in one-one correspondence with the complement of the variety $\det \sigma = 0$ in $E^{n(n+1)}$. The topology so induced characterizes G as a topological group, for the maps $G \times G \to G$ and $G \to G$ given by the operations of multiplication and taking inverses are continuous.

We shall use the absolute value symbol $|\sigma|$ to denote the Euclidean distance from the origin of the point of $E^{n(n+1)}$ corresponding to σ. Then $|\sigma|$ is unaltered by left or right multiplication by an element of O.

A closed set $\sum \subset G$ is compact if

$$\text{(i)} \qquad \inf_{\sigma \in \Sigma} |\det \sigma| > 0,$$

$$\text{(ii)} \qquad \sup_{\sigma \in \Sigma} |\sigma| < \infty,$$

for then the corresponding set in $E^{n(n+1)}$ is bounded and closed.

LEMMA 1. *If K is a fixed convex body, the mapping of G into $\mathfrak{C}$ given by*

$$\sigma \to \sigma K$$

is continuous.

Proof. Let $R > 1$ be an upper bound for the distance of points of τK from the origin, where τ is fixed. Then, if $x \in \tau K$ and e denotes the identity element of G,

$$|(\sigma x)_i - x_i| = |((\sigma - e)x)_i| \leqslant |\sigma - e|R(n+1).$$

Hence

$$\eta(\sigma\tau K, \tau K) \leqslant R|\sigma - e|n(n+1) \to 0$$

as $\sigma \to e$. This proves the lemma.

In the following two lemmas S denotes the solid unit sphere with its centre at the origin, and the A are positive constants.

LEMMA 2. *The set $\sum$ of $\sigma \in G$ satisfying the conditions*

$$S \supset \sigma S; \quad \det \sigma \geqslant A$$

is compact.

Proof. It follows from Lemma 1 that the set of σ satisfying the first of these two conditions is closed. Since the set $\det \sigma \geqslant A$ is obviously closed, $\sum$ is closed. Now the first of our compactness conditions is satisfied by hypothesis, so it is enough to show that $\sum$ corresponds to a bounded set in $E^{n(n+1)}$.

By (1) we can write $\sigma = \alpha_1\delta\alpha_2$. Then, since $\alpha S = S$ for all $\alpha \in O$,

$$S = \alpha_1^{-1}S \supset \alpha_1^{-1}\sigma S = \delta S.$$

Apply this to the points $(0, \ldots, 0)$ and $(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}})$. We find

$$\sum_{i=1}^{n} \delta_{i0}^2 \leqslant 1, \qquad \sum_{i=1}^{n} (n^{-\frac{1}{2}}\delta_{ii} + \delta_{i0})^2 \leqslant 1.$$

From the triangle inequality, $\sum \delta_{ii}^2 \leqslant 4n$. Hence $|\delta|^2 \leqslant 1 + 4n$; so, since transformations of O do not alter lengths in $E^{n(n+1)}$, $|\sigma|^2 \leqslant 1 + 4n$. This proves the lemma.

LEMMA 3. *Let K, K' be two convex bodies. Then the set $\sum_1$ of $\sigma \in G$ satisfying the conditions*

$$\sigma K' \supset K, \quad \det \sigma \leqslant A_2$$

is compact.

Proof. From Lemma 1, $\sum_1$ is closed. We shall show also that it is a subset of a compact set.

Since K has inner points $K \supset \tau_1 S$ for some $\tau_1 \in G$. Further K' is bounded, so $K' \subset \tau_2 S$. Hence the conditions defining $\sum_1$ imply

$$\sigma\tau_2 S \supset \tau_1 S, \qquad \det \sigma \leqslant A_2$$

or

$$S \supset \tau_2^{-1}\sigma^{-1}\tau_1 S, \qquad \det \sigma \leqslant A_2; \tag{2}$$

i.e., $\sum_1$ is contained in the set $\sum_2$ defined by the conditions (2). Apply to this set the homeomorphism $\sigma \to \tau_2^{-1}\sigma^{-1}\tau_1$. It follows from Lemma 2 that $\sum_2$ is compact. Hence $\sum_1$, being a closed subset of it, is also compact.

3. The invariant ρ. The proof of Theorem 1 depends on the introduction of a function ρ, defined as follows. Let K, K' be two convex bodies in E^n.

$$\rho(K,K') = \inf \frac{V(\sigma K')}{V(K)} \quad [\sigma \in G, \sigma K' \supset K],$$

where V denotes the content of the body.

By definition of ρ there is a sequence $\{\sigma_i\}$ of elements of G such that $\sigma_i K' \supset K$, $\frac{V(\sigma_i K')}{V(K)} \to \rho(K,K')$. Since $V(\sigma_i K') = |\det \sigma_i| V(K')$, $\det \sigma_i$ tends to a limit and so is bounded. Hence, by the compactness proved in Lemma 3, there is a subsequence such that $\lim \sigma_{i_\nu} = \tau \in G$; so by Lemma 1,

$$\tau K' \supset K, \quad \rho(K,K') = \frac{V(\tau K')}{V(K)}.$$

Thus $\rho(K,K')$, which by definition is not less than unity, is equal to unity only if K, K' belong to the same affine class.

Since affine transformations preserve ratios of content,

$$\rho(K, K') = \rho(\sigma K, \tau K') \qquad (\sigma, \tau \in G), \tag{3}$$

so ρ is really a function on $\mathfrak{C}^* \times \mathfrak{C}^*$. Next

$$\rho(K_1,K_3) \leqslant \rho(K_1,K_2)\,\rho(K_2,K_3). \tag{4}$$

For let $$\rho(K_1,K_2) = \frac{V(\sigma K_2)}{V(K_1)}\ ;\ \rho(K_2,K_3) = \frac{V(\tau K_3)}{V(K_2)}.$$

Then $$\rho(K_1,K_3) \leqslant \frac{V(\sigma\tau K_3)}{V(K_1)} = \frac{V(\tau K_3)}{V(K_2)}\,\frac{V(\sigma K_2)}{V(K_1)}.$$

LEMMA 4. *The function $\rho(K,K')$ is continuous.*

Proof. We prove first that, for each fixed K, $\rho(K,L) \to 1$ as $L \to K$.

Since K has inner points it contains a sphere, say centre O and radius r. Then if $\eta(K,L) < \frac{1}{2}r$, L contains a sphere centre O, radius $\frac{1}{2}r$. Let $(1+t)L$ be the body obtained by expanding L homothetically about O in the ratio $(1+t):1$. Then $(1+t)L$ is an affine image of L and contains all the points at distance $\frac{1}{2}tr$ or less from L; so, if $\epsilon < 1$,

$$\eta(K,L) < \tfrac{1}{2}\epsilon r \to (1+\epsilon)L \supset K$$

$$\to \rho(K,L) \leqslant V((1+\epsilon)L) : V(K) = (1+\epsilon)^n \frac{V(L)}{V(K)}$$

and the result follows from continuity of the content [2, p. 61, 62; 1, p. 38]. A similar argument shows that $\rho(L,K) \to 1$ as $L \to K$.

Now from (4) we have the relations

$$\rho(K'_1, K'_2) \leqslant \rho(K'_1, K_1)\, \rho(K_1, K_2)\, \rho(K_2, K'_2)$$

$$\rho(K_1, K_2) \leqslant \rho(K_1, K'_1)\, \rho(K'_1, K'_2)\, \rho(K'_2, K_2).$$

On letting $K'_1 \to K_1$, $K'_2 \to K_2$, we find that $\lim \rho(K'_1, K'_2) = \rho(K_1, K_2)$ and continuity is established.

LEMMA 5. *There is an absolute constant $c(n)$, depending only on the dimension of the space, such that*

$$\rho(K,K') \leqslant c(n)$$

for every two bodies K,K'.

Proof. Consider a simplex S of maximum content contained in K. Suppose its vertices are $A_0, \ldots, A_n$ and its faces $a_0, \ldots, a_n$. Let $b_0, \ldots, b_n$ be the hyperplanes through $A_0, \ldots, A_n$ respectively parallel to the opposite faces. These are hyperplanes of support of K, for if, say, b_0 were not, there would be points of K further from the hyperplane a_0 than A_0. If A' were such a point, then $A', A_1, \ldots, A_n$ would form a simplex of larger content, contradicting the original choice of S. Hence K is contained in the simplex $\sum$ defined by the hyperplanes $b_0, \ldots, b_n$. Now $V(\sum):V(S) = n^n$. Since all simplexes belong to the same affine class, we have

$$\rho(S,K)\, \rho(K,S) \leqslant n^n, \tag{5}$$

$$\rho(K,S) \leqslant n^n \text{ and } \rho(S,K') \leqslant n^n; \tag{6}$$

thus, from (4),

$$\rho(K,K') \leqslant n^{2n}. \tag{7}$$

It is now easy to prove the compactness part of Theorem 1. It follows from (6) that if S is a fixed simplex of unit content there is, for each K, a σK contained in S such that $V(\sigma K) \geqslant n^{-n}$. Then let $\mathfrak{B}$ be the set of all closed convex subsets of S which have content not less than n^{-n}. $\mathfrak{B}$ is compact, by Blaschke's theorem [1, p. 34; 2, p. 62-8] and by continuity of the content. Consider now the mapping of $\mathfrak{B}$ into $\mathfrak{C}^*$ where each body is mapped on the class of which it is a member. This map is continuous and the image is the whole of $\mathfrak{C}^*$; so, since $\mathfrak{B}$ is compact, its image $\mathfrak{C}^*$ is compact.[1]

[1]The condition $V(K) \geqslant n^{-n}$ is necessary only because we restrict ourselves to convex sets with inner points. This is the only interesting case, for if an affine invariant is continuous for all closed convex sets it is a constant. Thus if f is such a function and I denotes a closed interval, I can be derived from K by a singular affine map, and is the limit of a sequence of sets which are derived from K by non-singular affine maps. Hence $f(K) = f(I)$ and f is a constant.

From (4) it follows that the function

$$\Delta(K,K') = \log \rho(K,K') + \log \rho(K',K)$$

satisfies the triangle inequality, so it characterizes the affine classes as a metric space $\mathfrak{C}^{**}$. Let φ be the mapping of $\mathfrak{C}^{*}$ onto $\mathfrak{C}^{**}$ which maps each class, considered as a member of $\mathfrak{C}^{*}$ into the same class, considered as a member of $\mathfrak{C}^{**}$. Then, since Δ is a continuous function on $\mathfrak{C}^{*}$, the set of K such that

$$\Delta(K',K) < \epsilon \qquad (K' \text{ fixed})$$

is open in $\mathfrak{C}^{*}$. Since every open set in $\mathfrak{C}^{**}$ is a union of such sets, and so open in $\mathfrak{C}^{*}$, φ is a continuous map. Now $\mathfrak{C}^{*}$ is a compact space and $\mathfrak{C}^{**}$ is metric, so φ is topological; i.e. Δ gives a metrization for the space $\mathfrak{C}^{*}$. This completes the proof of Theorem 1.

Among the inequalities (5), (6), (7), only (5) is best possible. Equality holds in (5) if K is a sphere, but equality never holds in (6), since, from (5)

$$\rho(S,K) = n^n \rightarrow \rho(K,S) = 1 \rightarrow K = \sigma S \rightarrow \rho(S,K) = 1,$$

a contradiction. Since $\mathfrak{C}^{*}$ is compact, the upper bound is attained and so must be less than n^n.

4. A better bound for ρ. The above simple proof that ρ that is bounded was suggested to the author in a conversation with Dr. Mahler. This section gives the author's original proof, which implies a better bound for ρ. The proofs of the first two lemmas were suggested by a referee, and are shorter than the author's original proofs.

LEMMA 6. *Let P, Q be two points of a convex body K in E^n, where the distance $|PQ| = l$. Let K' be the orthogonal projection of K on π, a hyperplane perpendicular to PQ. Let V' be the $(n-1)$-dimensional content of K'. Then*

$$V(K) \geqslant lV'/n.$$

Proof. Symmetrize with respect to the hyperplane π [2, p. 44; 1, p. 69 sqq.] Let K_1 be the body obtained in this way. K_1 is convex and $V(K_1) = V(K)$. The intersection of K_1 and π is the projection K'. To the points P, Q correspond two points P_1, Q_1 of K_1 and $P_1Q_1 = l$.

By convexity, K contains the two cones with common base K' and vertices P, Q, whose total content is

$$lV'/n.$$

LEMMA 7. *There is a parallelotope Π containing K such that*[3]

$$V(\Pi) \leqslant n!V(K).$$

Proof by induction on n. Let P, Q be two points of K with maximum dis-

[3] A similar result for *symmetrical* convex bodies is proved in [3, p. 97-8].

tance $= l$. The two normal hyperplanes to the line PQ at P,Q are hyperplanes of support of K. Let K' be the projection of K on one of these hyperplanes. By the induction assumption there is an $(n-1)$-dimensional parallelotope Π' containing K' for which

$$V'(\Pi') \leqslant (n-1)!V'(K'),$$

where V' denotes $(n-1)$-dimensional content. Now K is contained in the parallelotope Π with base K' and altitude l. Hence

$$V(\Pi) = lV'(\Pi') \leqslant (n-1)!lV'(K'),$$

so by Lemma 6,

$$V(\Pi) \leqslant n!V(K).$$

LEMMA 8. *There is a parallelotope* Π *contained in* K, *such that*

$$V(\Pi) \leqslant n^{-n}V(K).$$

Proof by induction on n. Assume, without loss of generality, that, among the hyperplanes $x_1 = \text{const.}$, $x_1 = 0$ is the one or one of those whose section with K has maximum $(n-1)$-dimensional content. If $A(\mathfrak{a})$ is the $(n-1)$-dimensional content of the section $x_1 = \mathfrak{a}$, then

$$V(K) = \int_{-q}^{p} A(u)\,du \leqslant (p+q)\,A(0), \tag{8}$$

where p, $-q$ are the max and min of x_1-coordinates of points of K.

Let R be the intersection of K with the hyperplane $x_1 = 0$. By convexity, K contains the cones obtained by joining R to the two points P, Q that have max and min x_1-coordinate respectively. The hyperplanes $x_1 = p/n$, $-q/n$ intersect these cones in two congruent and similarly placed $(n-1)$-dimensional convex sets, each homothetic to R and each of $(n-1)$-content

$$A(0)\,(1-1/n)^{n-1}.$$

By the induction assumption, we can inscribe in these regions two congruent and similarly placed $(n-1)$-parallelotopes, each of content at least $A(0)/n^{n-1}$; and these, being distant $(p+q)/n$ apart, span an n-parallelotope of content at least

$$A(0)\,(p+q)/n^n \geqslant n^{-n}V(K)$$

by 8. This proves Lemma 8.

Lemmas 7, 8 combined with (4) give the sharper inequality

$$c(n) \leqslant n!\,n^n. \tag{9}$$

However, it is easy to convince oneself that neither Lemma 7 nor Lemma 8 is best possible, except Lemma 7 for $n = 2$. (Then equality holds if K is a triangle.) The problem of finding the exact value of $c(n)$ (which must be attained, by compactness, for some K, K') is left open. It seems natural to conjecture that $c(n)$ is attained when K is a simplex and K' is an ellipsoid.

The author would like to record his gratitude to Dr. Mahler and Mr. J. F. Nash for illuminating discussions and criticisms.

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THE BESSEL POLYNOMIALS

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1. Krall and Frink [2] have recently considered in connection with certain solutions of the wave equation a system of polynomials $y_n(x)$, $(n = 0, 1, 2, \ldots)$, where y_n is defined as that polynomial solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + (2x + 2) \frac{dy}{dx} = n(n + 1)y \tag{1}$$

which is equal to unity when $x = 0$.

They note the relationship of these polynomials to Hankel's functions of imaginary argument and establish among other results:

$$y_n = 2^{-n} e^{2/x} D^n (x^{2n} e^{-2/x}) \qquad \left(D = \frac{d}{dx}\right) \tag{2}$$

$$= \sum_{r=0}^{n} \frac{(n + r)!}{(n - r)!\, r!} \left(\frac{x}{2}\right)^r,$$

$$y_{n+1} = (2n + 1)x y_n + y_{n-1}, \tag{3}$$

$$\frac{1}{2\pi i} \int_C y_m y_n e^{-2/x} dx = \frac{(-)^{n+1} 2\epsilon_{mn}}{2n + 1}, \tag{4}$$

where C is the unit circle or any contour surrounding $x = 0$ and $\epsilon_{mn} = 0,1$ according as $m \neq n$, $m = n$.

It seems worthwhile to point out that the polynomials y_n are effectively the same as those encountered by T. W. Chaundy and the author in the course of a wider investigation [1]. Recognition of this fact leads to a more economical determination of the principal formulae of [2] as well as to other properties not mentioned by the authors of that paper. I therefore develop in more detail than was previously possible the properties of the polynomials in question.

2. It was shown in [1, pp. 478, 485] that the differential equation

$$\delta(\delta - 2n - 1)y = x^2 y \qquad \left(\delta = x \frac{d}{dx}\right), \tag{5}$$

where n is zero or a positive integer, has the solutions

$$y = \theta_n(x)e^{-x}, \theta_n(-x)e^{x},$$

where θ_n is a polynomial of degree n in x defined by

$$\theta_0 = 1, \theta_n = (-)^n e^x (\delta - 1)(\delta - 3) \ldots (\delta - 2n + 1)e^{-x}. \tag{6}$$

Received November 14, 1949.

We note that in θ_n the coefficient of x^n is unity and that more explicitly

$$\theta_n = \sum_{r=0}^{n} \frac{(n+r)!\,x^{n-r}}{2^r(n-r)!\,r!}. \tag{7}$$

A comparison of (7) with (2) shows that

$$y_n(x) = x^n\theta_n(1/x), \tag{8}$$

an identification which may be made without a knowledge of the explicit forms of θ_n, y_n by observing that, on setting $y = \theta_n e^{-z}$ in (5), we obtain

$$\delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n,$$

whence, without difficulty, $x^n\theta_n(1/x)$ is a solution of

$$\delta z + \tfrac{1}{2}x(\delta - n)(\delta + n + 1)z \tag{9}$$

which is the "δ" form of (1).

3. The zeros of θ_n, and so of y_n, have properties not mentioned in [2]. It was for instance shown in [1], as a corollary to a more general argument, that zeros a_r $(r = 1, 2, \ldots, n)$ of θ_n satisfy the relations

$$\sum_{r=1}^{n} a_r^{-1} = -1, \quad \sum_{r=1}^{n} a_r^{1-2s} = 0 \qquad (s = 2, 3, \ldots, n). \tag{10}$$

Hence the zeros b_r of y_n obey the relations

$$\sum_{r=1}^{n} b_r = -1, \quad \sum_{r=1}^{n} b_r^{2s-1} = 0 \qquad (s = 2, 3, \ldots, n). \tag{11}$$

An *ad hoc* proof of (10) is immediate: for let σ_k denote the sum of the kth powers of the zeros of θ_n and let $\theta_n = e^z\phi_n$, then

$$\frac{\phi'_n}{\phi_n} + 1 = \frac{\theta'_n}{\theta_n} = -\sum_{k=1}^{\infty} x^{k-1}\sigma_{-k}. \tag{12}$$

Reference to (6) shows that the expansion of ϕ_n in ascending powers contains no odd powers of x of index less than $2n + 1$. In consequence the expansion of ϕ'_n/ϕ_n contains no even powers of x with index less than $2n$ and so, on equating coefficients in (12), we have (10).

We may also establish the following results:

(13) *The polynomials θ_n, θ_{n+1} have no zero in common and no θ_n has a repeated zero.*

(14) *The polynomial θ_n has at most one real zero.*

Proofs of the statements in (13) are obtained by a familiar argument from the identities[1]

[1]It is worthy of notice that, if we define $\theta_{-n} = x^{1-2n}\theta_{n-1}$ the relations (15), (16) as well as the differential equation

$$\delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n$$

are also satisfied for negative integral n.

$$\theta'_n - \theta_n = -x\theta_{n-1}, \tag{15}$$

$$\theta_{n+1} - x^2\theta_{n-1} = (2n+1)\theta_n, \tag{16}$$

the latter of which is equivalent to (3).

To establish (15) we observe that

$$\begin{aligned} e^{-x}(\delta - x)\theta_n &= \delta\theta_n e^{-x} \\ &= (-)^n(\delta-3)(\delta-5)\dots(\delta-2n+1)\,\delta(\delta-1)e^{-x} \\ &= (-)^n(\delta-3)\dots(\delta-2n+1)x^2e^{-x} \\ &= (-)^n x^2(\delta-1)\dots(\delta-2n+3)e^{-x} \\ &= -x^2e^{-x}\theta_{n-1}. \end{aligned}$$

Again from (6)

$$(\delta - 2n - 1)\theta_n e^{-x} = -\theta_{n+1}e^{-x},$$

i.e.,

$$x(\theta'_n - \theta_n) = (2n+1)\theta_n - \theta_{n+1}$$

and, substituting on the left from (15), we have (16).

To prove (14) let

$$z_1 = e^{-x}\theta_n(x),\ z_2 = e^x\theta_n(-x),$$

then from (5),

$$z_1z'_2 - z_2z'_1 = Cx^{2n}$$

or,

$$\theta_n(-x)\theta'_n(x) + \theta_n(x)\theta'_n(-x) = 2\theta_n(x)\theta_n(-x) + Cx^{2n}, \tag{17}$$

where C is a constant shown by a simple calculation to be $2(-1)^{n+1}$. Now in $\theta_n(x)$ all coefficients are positive and so all real zeros are negative. If possible let $-\alpha$, $-\beta$ be two consecutive real zeros. Then, from (17),

$$\theta_n(\alpha)\theta'_n(-\alpha) = C\alpha^{2n},\ \ \theta_n(\beta)\theta'_n(-\beta) = C\beta^{2n}.$$

But $\theta_n(\alpha)$, $\theta_n(\beta)$ are both positive and so $\theta'_n(-\alpha)$, $\theta'_n(-\beta)$ have the same sign which is impossible. Hence θ_n has at most one real zero.

It is natural to enquire whether the zeros (b_r) of $y_n(x)$, in some order or other furnish the only solutions of the system of equations

$$\sum_{r=1}^{n} x_r = -1,\quad \sum_{r=1}^{n} x_r^{2s-1} = 0 \qquad (s = 2, 3, \dots, n).$$

This is in fact the case for, if (x_r) is any solution, let $-x_r = y_r$. Then

$$\sum_{r=1}^{n} b_r^{2s-1} + \sum_{r=1}^{n} y_r^{2s-1} = 0 \qquad (s = 1, 2, \dots, n).$$

The elementary symmetric functions A_{2s-1} of the numbers (b_r), (y_r) *taken together* therefore vanish when $s = 1, 2, \dots, n$ and the (b_r), (y_r) are the roots of an equation

$$t^{2n} + A_2t^{2n-2} + \dots + A_{2n} = 0,$$

containing only even powers of t. To every root t there corresponds a root $-t$. Now reference to (17) shows that no two zeros of θ_n can differ in sign only and the same is therefore true of the (b_r). Hence the (y_r) are the (b_r) in some order or other.

4. A pseudo-generating function for $\theta_n(x)$. I establish the formula

$$\frac{e^{2xu}}{1-2u} = \sum_{n=0}^{\infty} \frac{\theta_n(x)}{n!} [2u(1-u)]^n \tag{18}$$

for sufficiently small values[2] of u, from which the generating function for y_n and other variants in [2] may be derived. For the right-hand side of (18) is

$$\begin{aligned}
&e^x \sum_{n=0}^{\infty} (-)^n \frac{[2u(1-u)]^n}{n!} (\delta-1)(\delta-3)\dots(\delta-2n+1)\, e^{-x} \\
&\quad = e^x \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-)^n[2u(1-u)]^n}{n!} (m-1)\dots(m-2n+1) \frac{(-x)^m}{m!} \\
&\quad = e^x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-x)^m}{m!} \frac{(\frac{1}{2}-\frac{1}{2}m)_n}{n!} [4u(1-u)]^n \\
&\quad = e^x \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} (1-4u+4u^2)^{\frac{1}{2}m-\frac{1}{2}} \\
&\quad = \frac{e^x}{1-2u} \sum_{m=1}^{\infty} \frac{(-x)^m(1-2u)^m}{m!} \\
&\quad = \frac{e^{2xu}}{1-2u}.
\end{aligned}$$

5. The authors of [2] have also considered the equation

$$x^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} = n(n+a-1)y,$$

or

$$b\delta y + x(\delta+n+a-1)(\delta-n)y = 0,$$

with polynomial solutions $y_n(x,a,b)$ made definite by the supplementary condition[3] $y_n(0) = 1$. Defining $\phi_n(x,a,b) \equiv \phi_n$ by

$$\phi_n = x^n y_n(x^{-1},a,b),$$

we find that ϕ_n is a solution of

$$\delta(\delta+1-a-2n)z = bx(\delta-n)z, \tag{19}$$

[2] We may take conveniently $0 < u < \frac{1}{2}$.

[3] It is evident from the second form of the differential equation that the constant b is a mere scale-factor, and nothing would be lost by considering only $b = \pm 1$. I retain b for the sake of comparison with the formulae of [2].

and that $e^{-bx}\phi_n$ is a solution of

$$\delta(\delta + 1 - a - 2n)w = -bx(\delta - n - a + 2)w. \tag{20}$$

We note that, if $a = 2$, (19) and (20) differ only in the sign of b and that (20) will then have the solutions

$$e^{-bx}\phi_n(x,2,b), \quad \phi_n(x,2,-b)$$

leading us back effectively to the theory of §2.

We observe also that, if $n + a - 2$ is zero or a positive integer, (20) will have a polynomial solution of that degree. I now show that *equation* (20) *has the solution*

$$w = (\delta - n - a + 1)(\delta - n - a)\dots(\delta - 2n - a + 2)e^{-bx}. \tag{21}$$

For

$$\begin{aligned}\delta(\delta - 2n - a + 1)w &= (\delta - n - a + 1)\dots(\delta - 2n - a + 1)\delta e^{-bx}\\ &= (\delta - n - a + 1)\dots(\delta - 2n - a + 1)(-bx)e^{-bx}\\ &= -bx(\delta - n - a + 2)\dots(\delta - 2n - a + 2)e^{-bx}\\ &= -bx(\delta - n - a + 2)w.\end{aligned}$$

Recalling that the coefficient of x^n in ϕ_n is unity we have

$$\begin{aligned}\phi_n(x,a,b) &= (-b)^{-n}e^{bx}(\delta - n - a + 1)\dots(\delta - 2n - a + 2)e^{-bx}\\ &= (-b)^{-n}e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}),\end{aligned} \tag{22}$$

the latter form being equivalent to (47) of [2] and to (2) of the present note on setting $a = b = 2$. Thus, in addition to the formula (6) for the θ_n of §2, we have

$$\begin{aligned}e^{-2x}\theta_n(x) &= (-\tfrac{1}{2})^n(\delta - n - 1)\dots(\delta - 2n)e^{-2x}\\ &= (-\tfrac{1}{2})^n x^{2n+1}D^n(x^{-n-1}e^{-2x}).\end{aligned} \tag{23}$$

6. The operational methods of the present note enable me to repair one omission in [2], namely the failure to supply a generating function for the polynomials $y_n(x,a,b)$. I establish in the first place the result

$$\frac{(1-u)^{2-a}e^{bxu}}{1-2u} = \sum_{n=0}^{\infty}\frac{[bu(1-u)]^n\phi_n}{n!}, \tag{24}$$

which reduces to (18) when $a = b = 2$. We require the auxiliary formula

$$\sum_{n=0}^{\infty}\frac{(n-k)_n[u(1-u)]^n}{n!} = (1-2u)^{-1}(1-u)^{k+1}, \tag{25}$$

which may be proved as follows.

The coefficient of u^s in

$$\sum_{n=0}^{\infty}\frac{(n-k)_n u^n(1-u)^{n-1-k}}{n!} \quad \text{is} \quad \sum_{n=0}^{s}\frac{(-)^{s-n}(2n-k-s)_s}{n!(s-n)!}$$

and this is the coefficient of t^s in the expansion of

$$\sum_{n=0}^{s} \frac{(-)^n s!}{n!(s-n)!}(1+t)^{k+s-2n} = (1+t)^{k+s}[1-(1+t)^{-2}]^s = (1+t)^{k-s}t^s(2+t)^s$$

in which the coefficient of t^s is 2^s.

Returning now to (24), on the right we have, by (22) and (25),

$$e^{bx}\sum_{n=0}^{\infty}\frac{(-)^n[u(1-u)]^n}{n!}\sum_{m=0}^{\infty}\frac{(m-2n-a+2)_n(-bx)^m}{m!}$$

$$= e^{bx}\sum_{m=0}^{\infty}\frac{(-bx)^m}{m!}\sum_{n=0}^{\infty}\frac{(n+a-1-m)_n[u(1-u)]^n}{n!}$$

$$= e^{bx}(1-2u)^{-1}\sum_{m=0}^{\infty}\frac{(-bx)^m(1-u)^{m+2-a}}{m!}$$

$$= (1-2u)^{-1}(1-u)^{2-a}e^{bxu},$$

as required. Writing x^{-1} for x and xu for u in (24), we have

$$\frac{(1-xu)^{2-a}e^{bu}}{1-2xu} = \sum_{n=0}^{\infty}\frac{b^n y_n(x,a,b)[u(1-xu)]^n}{n!} \tag{26}$$

and, setting

$$2u(1-xu) = t, \text{ or } 2xu = 1-(1-2xt)^{\frac{1}{2}}$$

in (26), we find

$$[\tfrac{1}{2}+\tfrac{1}{2}(1-2xt)^{\frac{1}{2}}]^{2-a}(1-2xt)^{-\frac{1}{2}}\exp\left[\frac{b}{2x}\{1-(1-2xt)^{\frac{1}{2}}\}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(b/2)^n y_n(x,a,b)t^n}{n!},$$

which may serve as a generating function for the polynomials $y_n(x,a,b)$.

7. In this section I assume that a is a positive integer. The results obtained may in certain circumstances be extended to negative integral and zero values of a but at the cost of their ceasing to hold for all n.

In the first place we observe that, when a is a positive integer (20) has a polynomial solution of degree $n+a-2$ and a series solution in ascending powers of x led by x^{2n+a-1}. On the other hand we see that the expansion of the w defined by (21) lacks all terms with indices between $n+a-1$ and $2n+a-2$ inclusive. The expansion may in fact be divided by this gap into two parts furnishing respectively a polynomial and a series solution. A more important consequence of this gap in the expansion of w is the following. From (21) we have

$$e^{-bx}\phi_n = \sum_{r=0}^{a+n-2} c_r x^r + \sum_{r=-1}^{\infty} c_{2n+a+r}x^{2n+a+r}$$

where

$$c_0 = b^{-n}(n+a-1)_n,\ c_{2n+a-1} = (-b)^{n+a-1}n!/(2n+a-1)!.$$

Suppose now $m < n$; then in the expansion of $e^{-bx}\phi_m\phi_n$ the term in $x^{m+n+a-1}$ is missing, while in the expansion of $e^{-bx}\phi_n{}^2$ the coefficient of x^{2n+a-1} is c_0c_{2n+a-1}. Hence, if C is any contour surrounding $x = 0$,

$$\frac{1}{2\pi i}\int_C \frac{\phi_m\phi_n e^{-bx}}{x^{m+n+a}}\,dx = \frac{\epsilon_{mn}(-)^{n+a-1}b^{a-1}n!}{(2n+a-1)(n+a-2)!} \tag{28}$$

and, on changing the variable to $1/x$,

$$\frac{1}{2\pi i}\int_C \frac{y_m(x,a,b)y_n(x,a,b)e^{-b/x}}{x^{2-a}}\,dx = \frac{\epsilon_{mn}(-)^{n+a-1}b^{a-1}n!}{(2n+a-1)(n+a-2)!} \tag{29}$$

which reduces to (4) when $a = b = 2$. The problem of an appropriate weight function when a is not integral has been considered in [2], and to that discussion I have nothing to add.

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VORTICITY AVERAGES

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1. Introduction. Recent studies of turbulent fluid motions have drawn attention to the transfer of vorticity. There have been some attempts to study turbulence by plane models, but these have been criticized justly for failing to reveal the true nature of a phenomenon which depends essentially on three-dimensional convection. I have thought it worthwhile to eschew current conjectures regarding turbulence, turning in preference to the method of a master of the theory of vortices and applying it to the discovery of certain average properties of the continuous rotational motion of any medium whatever. This memoir therefore makes no attempt to deal with the problem of turbulence, but it is possible that the theorems presented here, being exact consequences of the kinematical equations, may nevertheless enjoy a certain relevance.

This method was introduced originally to obtain conservation theorems valid in a class of plane motions of an inviscid fluid or of a viscous incompressible fluid. Some time ago the results were substantially generalized in a purely kinematical form, but still subject to the apparently essential but hardly desirable restriction to plane isochoric motion. In this paper I shall show that a somewhat different sequence of conservation theorems, expressed in terms of

Received December 16, 1949. The author is grateful to his colleague Dr. Neményi for drawing his attention to the total vorticity in 1946, and for constructive criticism of this memoir.

symmetrical vector moments of the vorticity vector, is appropriate to the three-dimensional motion of an arbitrary continuous medium.

It is interesting to discover that in one respect at least three-dimensional motions are simpler in their average properties than are plane motions. The reason lies partly in the different and apparently more complicated mechanism of convection in the three-dimensional case, but more essentially in the fact that *only in a space of three dimensions is the vorticity a vector*. So as to make use of the various cross operations which exist only in three-dimensional space, and which are essential in our subject, I shall employ the suggestive and elegant polyadic notations of Gibbs[1] [5].

In Parts I and II, the former of which is devoted to plane and the latter to three-dimensional motions, are collected all the vorticity integrals known up to the present time. These are presented in a generalized kinematical form, and compared and contrasted with one another. Part III contains two new general conservation theorems[2]. I hope that the relative shortness of this last portion will not be taken as indicative of its relative value, for in the present subject the generality and applicability of a result seems to vary in inverse ratio to the length and complexity of its proof.

Part I. Known Vorticity Integrals for Plane Motions

2. Poincaré's theory of vortices in plane flows of inviscid fluids. In the dynamics of systems of mass-points or of rigid bodies the construction of an integral of the motion represents substantial progress towards a complete solution of the problem. In the dynamics of continuous media, endowed with an infinite number of degrees of freedom, even an infinite number of integrals usually does not suffice for the solution of any specific problem, yet from the earliest days of continuum mechanics such integrals have been sought, sometimes because they display kinematical or physical properties of the medium affording some insight and grasp, sometimes because they enable the transformation of one problem into another or the construction of analogies between different disciplines, and sometimes for their mathematical elegance. The second of these possibilities suggested to Poincaré [18, §§65, 75, 123] that since the vorticity

$$w = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{2.1}$$

of a plane motion of an inviscid incompressible fluid subject to conservative extraneous force remains constant for each particle, it might well be taken as the analogue of the mass-density in rigid dynamics. Introducing the "mass"

[1] My only departure from Gibbsian notations is to replace ∇, $\nabla\,\cdot$, and $\nabla\times$, by grad, div, curl, respectively, and to use boldface letters according to the convention customary in works in mathematical physics.

[2] The first of these has been announced in [25].

$$\mathfrak{W} = \int_{\mathfrak{S}} d\mathfrak{S}w, \tag{2.2}$$

the "centre of mass" $\mathbf{r}_w$:

$$\mathfrak{W}\mathbf{r}_w = \int_{\mathfrak{S}} d\mathfrak{S}w\mathbf{r}, \tag{2.3}$$

where $\mathbf{r}$ is the radius vector, and the "moment of inertia"

$$\mathfrak{I} = \int_{\mathfrak{S}} d\mathfrak{S}wr^2, \tag{2.4}$$

Poincaré proved that, subject to suitable conditions of regularity of the velocity field, when the area occupied by the fluid is the whole plane these quantities are constant in time, and thus provide integrals of the motion. In the special case of an irrotational motion induced by vortices the integrals are to be replaced by finite sums over the vortices, and the conservation theorems are still valid, but now Poincaré obtained for the motion of the vortices equations of the same form as Hamilton's equations for mass-points, so that the integrals represent substantial progress towards the complete integration, and indeed when there are but three vortices present the problem is reduced to quadratures [18, §77].

3. Hamel's integrals. There are two ways in which one might seek to generalize Poincaré's results, still considering only plane motions. First, one might try to extend the validity of his integrals to more general media. Second, one might attempt to form other quantities

$$\mathfrak{K} = \int_{\mathfrak{S}} d\mathfrak{S}w\sum a_{ij}x^i y^j \tag{3.1}$$

of polynomial moment type, or perhaps still more general expressions, which are constant in time. Poincaré himself followed the first course, investigating the behaviour of his three integrals in motions of homogeneous viscous incompressible fluids filling the whole plane. He showed that the "mass" and "centre of mass" remain constant, while the "moment of inertia" satisfies the equation

$$\frac{d\mathfrak{I}}{dt} = 4\nu\mathfrak{W}, \tag{3.2}$$

where ν is the kinematic viscosity [18, §§154-156]. A result in accord with the second course was obtained incidentally in a different sort of investigation by Hamel[3] [6]; it may be expressed in the following way:

[3] I learned of (3.3) through its rediscovery by Kampé de Feriet [8, 9]. For the reference to Hamel's work I am indebted to Professor Weinstein and to the referee; the former has called my attention also to a paper by Zaremba [29], where the result is apparently employed without explicit statement, and to the proofs of the more difficult converse by Weyl [28] and himself [27]. All these authors state the result in terms of potential theory, and only Kampé de Feriet observes that it can be interpreted as the vorticity theorem above.

Let w be the vorticity of a continuously differentiable plane isochoric[4] *motion such that upon the boundary $\mathfrak{C}$ of a finite domain $\mathfrak{S}$ the material adheres without slipping; then if ϕ be a function harmonic in $\mathfrak{S}$,*

$$\int_{\mathfrak{S}} d\mathfrak{S} w\phi = 0. \tag{3.3}$$

4. Comparison and contrast of the integrals of Poincaré and Hamel: their limitations. In the special cases $\phi = 1, x, y$ Hamel's integrals reduce to Poincaré's "mass" and to the products of the "mass" by the components of the "centre of mass", respectively, but $\phi = x^2 + y^2$ is not a harmonic function, and from Hamel's theorem we obtain

$$\int_{\mathfrak{S}} d\mathfrak{S} w(x^2 - y^2) = 0, \qquad \int_{\mathfrak{S}} d\mathfrak{S} wxy = 0, \tag{4.1}$$

in place of Poincaré's "moment of inertia" integral. In fact Kampé de Feriet [8] notes that Hamel's theorem yields two linear relations connecting the various special moments of type (3.1) with $i + j = n$, for any n.

Poincaré's results apply to motions filling the whole plane while Hamel's apply to motions within a finite area. This distinction is not essential, since the results of either author are easily generalized to motions bounded in part by finite walls to which the material adheres without slipping and in part unbounded, subject to certain conditions on the manner in which the motion must vanish at infinity. The second apparent difference, that Poincaré's results concern the constancy of certain integrals while Hamel's concern the vanishing of certain integrals, is also inessential, for the method of both authors—indeed it is the method of Poincaré's whole theory of vortices—is to transform a surface integral into a line integral and then state conditions strong enough to secure the vanishing of this latter quantity. Now if such a transformation exist for a given integral, there is also necessarily a similar transformation for its time rate of change, which consequently can be shown to vanish under somewhat weaker conditions.

While Hamel's theorem is most elegant, I cannot help feeling that somehow it fails to reveal the essential average properties of vorticity because it is restricted to isochoric motions[5], which represent an extreme idealization. Now in view of Ampère's transformation

$$\int_{\mathfrak{S}} d\mathfrak{S} w = \oint_{\mathfrak{C}} (v_x dx + v_y dy), \tag{4.2}$$

the special case $\phi = 1$ is not so restricted, but is valid for any continuously differentiable motion. In the course of this investigation I was unable to rid

[4] An isochoric motion is one in which material volumes are preserved: div $\mathbf{v} = 0$.

[5] In Hamel's treatment this restriction arises through the consideration of the Laplacian of the stream-function, which is proportional to the vorticity only in isochoric motions, and indeed a single stream-function does not necessarily exist in the more general case.

these theorems of this unfortunate limitation, so long as plane motion be the subject of enquiry.

This point has now been clarified by Synge [31], who has shown that Hamel's theorem is a special case of a *characterization* of plane fields defined within and vanishing upon a single closed boundary, the general result no longer yielding a simple vorticity theorem since it depends (as would be expected) upon the expansion θ as well. I have constructed two analogous characterizations [32, 33] for the three-dimensional case, again in terms of integrals whose integrands contain both $\mathbf{w}$ and θ, integrals which in the special case $\theta = 0$ reduce to the theorems IV and I of Berker, respectively, given as formulae (9.14) and (6.10) below. The present memoir does not seek characterizations, aiming rather to formulate conditions sufficient that certain quantities be integrals of the motion.

Part II. Known Vorticity Integrals for Three-dimensional Motions

5. Plan of the analysis of three-dimensional motions. In a three-dimensional motion the vorticity

$$\mathbf{w} \equiv \operatorname{curl} \mathbf{v} \tag{5.1}$$

is an axial vector. Before Poincaré's study of the vorticity of plane motions, Lamb had demonstrated the vanishing of a volume integral of $\mathbf{w}$, but it involves a product of $\mathbf{w}$ by $\mathbf{v}$ rather than by $\mathbf{r}$, and it applies only to isochroric motions. Poincaré himself exhibited a second integral of this type. Both these results, and their recent generalization by Berker, are discussed in §6. If we were to proceed by analogy to Poincaré's treatment of the plane case we might introduce and analyze such moments as

$$\int_{\mathfrak{B}} \mathbf{w} d\mathfrak{B}, \qquad \int_{\mathfrak{B}} \mathbf{r} \cdot \mathbf{w} d\mathfrak{B}, \qquad \int_{\mathfrak{B}} \mathbf{r} \times \mathbf{w} d\mathfrak{B}, \qquad \int_{\mathfrak{B}} r^2 \mathbf{w} d\mathfrak{B}.$$

From results of Moreau published a few months ago it is shown in §8 that the third and fourth of these moments are integrals of the motion only in case it is circulation-preserving and isochoric. The first, however, was shown to be an integral under rather general circumstances by A. Föppl and Jaffé, and a year ago Berker showed that the second also is a possible integral; these results are outlined in §9.

The foregoing rather limited gleaning might suggest that as in plane motion so also in the three-dimensional case the search for vorticity integrals quickly reaches a point of diminishing returns. In fact, however, all previous investigators of this subject have cultivated an unwilling soil. By planting the fresh seed of *symmetrical vector moments* of a vector we shall find in §§11-12 that Poincaré's method of reduction—in this case, reduction of a volume integral to a surface integral, which vanishes subject to certain simple conditions —is always applicable, and will enable us to reap an unexpectedly rich harvest

of integrals. Thus it will appear that Berker's scalar moment may be an integral because under similar circumstances the symmetrical vector moment

$$\int_{\mathfrak{B}}(\mathbf{rw}+\mathbf{wr})d\mathfrak{V},$$

of which it is the scalar, may be an integral. On the other hand, Moreau's second scalar moment cannot be expected to be an integral in general, since the symmetrical second vector moment

$$\int_{\mathfrak{B}}(\mathbf{rrw}+\mathbf{rwr}+\mathbf{wrr})d\mathfrak{V}$$

may well be an integral, whence by contraction arises not Moreau's moment but rather

$$\int_{\mathfrak{B}}(r^2\mathbf{w}+2\mathbf{r}\cdot\mathbf{wr})d\mathfrak{V}.$$

6. The integrals of Lamb, Poincaré, and Berker for isochoric motions. The analysis of this section rests upon two easily demonstrated forms of Green's transformation[6]:

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot(\mathbf{bc}+\mathbf{cb})-d\mathfrak{S}\mathbf{b}\cdot\mathbf{c}]$$

$$(6.1)\qquad =\int_{\mathfrak{B}}[\operatorname{curl}\mathbf{c}\times\mathbf{b}+\operatorname{curl}\mathbf{b}\times\mathbf{c}+\mathbf{b}\operatorname{div}\mathbf{c}+\mathbf{c}\operatorname{div}\mathbf{b}]d\mathfrak{V},$$

$$\oint_{\mathfrak{S}}[d\mathfrak{S}\cdot(\mathbf{bc}+\mathbf{cb})\mathbf{r}-d\mathfrak{S}\mathbf{b}\cdot\mathbf{cr}]$$

$$(6.2)\qquad =\int_{\mathfrak{B}}[\mathbf{bc}+\mathbf{cb}-\mathbf{c}\cdot\mathbf{b}\mathbf{I}+(\operatorname{curl}\mathbf{c}\times\mathbf{b}+\operatorname{curl}\mathbf{b}\times\mathbf{c}+\mathbf{b}\operatorname{div}\mathbf{c}+\mathbf{c}\operatorname{div}\mathbf{b})\mathbf{r}]d\mathfrak{V},$$

for whose validity it is sufficient that $\mathbf{b}$ and $\mathbf{c}$ be continuously differentiable within $\mathfrak{B}$ and continuous upon $\mathfrak{S}$.

In (6.1) put $\mathbf{b}=\mathbf{c}=\mathbf{v}$ and suppose $\operatorname{div}\mathbf{v}=0$; there results

$$(6.3)\qquad \oint_{\mathfrak{S}}[d\mathfrak{S}\cdot\mathbf{vv}-d\mathfrak{S}\tfrac{1}{2}v^2]=\int_{\mathfrak{B}}\mathbf{w}\times\mathbf{v}d\mathfrak{V}.$$

By formulating conditions under which the surface integral vanishes, we obtain the integral of Lamb [12, 13]:

In a continuously differentiable isochoric motion, if the material adhere without slipping ($\mathbf{v}=0$) *upon all finite boundaries, while in any portion of the material extending to infinity the condition* $rv\to 0$ *as* $r\to\infty$ *be satisfied, then*

$$(6.4)\qquad \int_{\mathfrak{B}}\mathbf{w}\times\mathbf{v}d\mathfrak{V}=0.$$

[6] In [25, 26] it is shown that these two transformations are the cases $n=0$ and $n=1$ of a new general form of Green's transformation containing an arbitrary positive integer n.

Notice that while this theorem applies to a wide class of motions of viscous incompressible fluids, including all regular motions in a bounded domain of any connectivity, since the adherence condition is not satisfied in the absence of friction the result can hold for motions of incompressible perfect fluids only in case they fill all of space.

In (6.2) put $\mathbf{b} = \mathbf{c} = \mathbf{v}$ and suppose div $\mathbf{v} = 0$; there results

$$\oint_{\mathcal{S}}[d\mathcal{S}\cdot\mathbf{vv}\mathbf{r} - d\mathcal{S}\mathbf{r}\tfrac{1}{2}v^2] = \int_{\mathcal{V}}[\mathbf{vv} - \tfrac{1}{2}v^2\mathbf{I} + \mathbf{w}\times\mathbf{v}\mathbf{r}]d\mathcal{V}. \tag{6.5}$$

Taking the scalar of this result yields the kinetic energy formula of Lamb[7] [12, 13] and J. J. Thomson [21]:

$$\tfrac{1}{2}\int_{\mathcal{V}}v^2 d\mathcal{V} = \oint_{\mathcal{S}}d\mathcal{S}\cdot(\tfrac{1}{2}v^2\mathbf{r} - \mathbf{vv}\cdot\mathbf{r}) + \int_{\mathcal{V}}\mathbf{r}\cdot\mathbf{w}\times\mathbf{v}d\mathcal{V}, \tag{6.6}$$

while taking the vector yields

$$\oint_{\mathcal{S}}[d\mathcal{S}\cdot\mathbf{vr}\times\mathbf{v} + d\mathcal{S}\times\mathbf{r}\tfrac{1}{2}v^2] = \int_{\mathcal{V}}\mathbf{r}\times(\mathbf{w}\times\mathbf{v})d\mathcal{V}, \tag{6.7}$$

whence we obtain the integral of Poincaré [18, §115]:

In a continuously differentiable isochoric motion, if the material adhere without slipping ($\mathbf{v} = 0$) *upon all finite boundaries, while in any portion of the material extending to infinity the condition* $r^3v^2 \to 0$ *as* $r \to \infty$ *be satisfied, then*

$$\int_{\mathcal{V}}\mathbf{r}\times(\mathbf{w}\times\mathbf{v})d\mathcal{V} = 0. \tag{6.8}$$

Notice the conditions for the validity of Poincaré's integral insure the simultaneous validity of Lamb's integral.

In (6.1) put $\mathbf{c} = \mathbf{v}$ and suppose div $\mathbf{v} = 0$, and suppose further that curl $\mathbf{b} = 0$, div $\mathbf{b} = 0$ (i.e. $\mathbf{b} = \text{grad}\ \phi$, where $\nabla^2\phi = 0$); there results

$$\oint_{\mathcal{S}}[d\mathcal{S}\cdot(\mathbf{bv} + \mathbf{vb}) - d\mathcal{S}\mathbf{v}\cdot\mathbf{b}] = \int_{\mathcal{V}}\mathbf{w}\times\mathbf{b}d\mathcal{V}, \tag{6.9}$$

whence we obtain the theorem IV of Berker [2, 30]:

Given a continuously differentiable isochoric motion such that upon all finite boundaries the material adheres without slipping, then for any continuously differentiable harmonic vector $\mathbf{b}$ *we have*

$$\int_{\mathcal{V}}\mathbf{w}\times\mathbf{b}d\mathcal{V} = 0, \tag{6.10}$$

provided that in any portion of the material extending to infinity the condition $bvr^2 \to 0$ *be satisfied.*

[7] Notice that the conservation of the actual kinetic energy in a plane motion is equivalent to the conservation of the "moment of momentum" in the analogy; of [18, §76].

7. Three basic formulae of kinematics. It was shown by Euler [3] that $D \log d\mathfrak{V}/Dt = \operatorname{div} \mathbf{v}$. Hence if $\mathfrak{V}$ be a finite material volume and if Σ be any polyadic, subject to the usual conditions for differentiating an integral we have

$$\frac{D}{Dt}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} = \int_{\mathfrak{V}} \left(\frac{D\Sigma}{Dt} + \Sigma \operatorname{div} \mathbf{v}\right) d\mathfrak{V}. \tag{7.1}$$

By an application of Green's transformation to (7.1) we obtain the *transport theorem* of Reynolds[8] [19]:

$$\frac{D}{Dt}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} = \frac{\partial}{\partial t}\int_{\mathfrak{V}} \Sigma\, d\mathfrak{V} + \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{v}\Sigma. \tag{7.2}$$

By calculating the curl of the acceleration it is easy to deduce the vorticity equation of Lagrange and Beltrami[9]:

$$\frac{D\mathbf{w}}{Dt} = \operatorname{curl} \mathbf{a} + \mathbf{w} \cdot \operatorname{grad} \mathbf{v} - \mathbf{w} \operatorname{div} \mathbf{v}. \tag{7.3}$$

In the results of the succeeding sections two boundary conditions are employed. First, upon any stationary boundary

$$d\mathfrak{S} \cdot \mathbf{v} = 0. \tag{7.4}$$

Second, upon a stationary boundary to which the material adheres without slipping ($\mathbf{v} = 0$), the vorticity is wholly tangential [14]:

$$d\mathfrak{S} \cdot \mathbf{w} = 0. \tag{7.5}$$

In what follows let b_r denote the radial component of $\mathbf{b}$, let $\mathbf{b}_t$ denote the projection of $\mathbf{b}$ onto a plane perpendicular to $\mathbf{r}$, and let b_t denote the magnitude of $\mathbf{b}_t$. Similarly, for a dyadic Σ, by Σ_r we shall mean its projection in the radial direction, and by $|\Sigma_r|$ the magnitude of that projection.

8. The integrals of Moreau for isochoric circulation-preserving motions. Let us decompose the acceleration $\mathbf{a}$ into a "conservative" portion grad W and a "non-conservative" portion $\mathbf{a}'$:

$$\mathbf{a} = \operatorname{grad} W + \mathbf{a}'. \tag{8.1}$$

Such a resolution is possible in an infinite number of ways, and of itself has no kinematical significance; in a dynamical application, however, some particular decomposition of this type often has special meaning. In any case curl $\mathbf{a}$ =

[8] Curiously enough Reynolds saw fit to lay down this easily demonstrated transformation as a postulate in his theory of matter; still more curiously, in the literature of physics it is often applied tacitly in special cases as if it were "obvious" (i.e., too difficult to prove). The only text book of vector analysis in which I have found it mentioned is [20].

[9] The case curl $\mathbf{a} = 0$, which in the still more special case div $\mathbf{v} = 0$ is generally called "Helmholtz's equation", was first derived by Lagrange [10]. The equation (7.3) is given by Beltrami [1].

curl $\mathbf{a}'$. If there exist such a decomposition with $\mathbf{a}' = 0$, Kelvin's theorem of circulation holds and we may call the motion a *circulation-preserving motion.* Moreau [15, 16] has recently discussed the dynamics of isochoric motions by a method equivalent to finding the resultant and vector moment of $\mathbf{a}'$, which we shall now derive by a purely kinematical analysis.

By combining the three vectorial identities

$$\operatorname{div} \mathbf{r}\mathbf{a}' = 3\mathbf{a}' + \mathbf{r}\cdot\operatorname{grad} \mathbf{a}', \tag{8.2}$$

$$\operatorname{grad} (\mathbf{r}\cdot\mathbf{a}') = \mathbf{r}\cdot\operatorname{grad} \mathbf{a}' + \mathbf{a}' + \mathbf{r} \times \operatorname{curl} \mathbf{a}', \tag{8.3}$$

$$\operatorname{div} (\mathbf{w}\mathbf{r} \times \mathbf{v}) = \mathbf{w} \times \mathbf{v} + \mathbf{r} \times (\mathbf{w}\cdot\operatorname{grad} \mathbf{v}), \tag{8.4}$$

for any motion we have

$$\begin{aligned} &\mathbf{r} \times [\operatorname{curl} \mathbf{a}' + \mathbf{w}\cdot\operatorname{grad} \mathbf{v}] \\ &\quad = \mathbf{v} \times \mathbf{w} + 2\mathbf{a}' + \operatorname{grad} (\mathbf{r}\cdot\mathbf{a}') + \operatorname{div} (\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}'). \end{aligned} \tag{8.5}$$

Since curl $\mathbf{a}' = \operatorname{curl} \mathbf{a}$, by Euler's transformation (7.1), the Lagrange-Beltrami equation (7.3), and Green's transformation we thus obtain

$$\begin{aligned} \frac{D}{Dt}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} &= \int_{\mathfrak{V}} \left[\mathbf{v} \times \mathbf{w} + \mathbf{r} \times \left(\frac{D\mathbf{w}}{Dt} + \mathbf{w} \operatorname{div} \mathbf{v}\right)\right] d\mathfrak{V}, \\ &= \int_{\mathfrak{V}} [\mathbf{v} \times \mathbf{w} + \mathbf{r} \times (\operatorname{curl} \mathbf{a}' + \mathbf{w}\cdot\operatorname{grad} \mathbf{v})] d\mathfrak{V}, \\ &= \int_{\mathfrak{V}} [2\mathbf{v} \times \mathbf{w} + 2\mathbf{a}'] d\mathfrak{V} + \oint_{\mathfrak{S}} [d\mathfrak{S}\mathbf{r}\cdot\mathbf{a}' + d\mathfrak{S}\cdot(\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}')]. \end{aligned} \tag{8.6}$$

For an isochoric motion we may express the first term on the right as a surface integral by (6.3):

$$\begin{aligned} \frac{D}{Dt}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{a}'\, d\mathfrak{V} + \oint_{\mathfrak{S}} [&d\mathfrak{S}(\mathbf{r}\cdot\mathbf{a}' + v^2) \\ &+ d\mathfrak{S}\cdot(-2v\mathbf{v} + \mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{r}\mathbf{a}')], \end{aligned} \tag{8.7}$$

hence by the transport theorem (7.2) of Reynolds obtaining finally

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w}\, d\mathfrak{V} = 2\int_{\mathfrak{V}} \mathbf{a}'\, d\mathfrak{V} + \oint_{\mathfrak{S}} [&d\mathfrak{S}(\mathbf{r}\cdot\mathbf{a}' + v^2) \\ &+ d\mathfrak{S}\cdot(-2v\mathbf{v} + \mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{v}\mathbf{r} \times \mathbf{w} - \mathbf{r}\mathbf{a}')]. \end{aligned} \tag{8.8}$$

By formulating conditions sufficient that the right-hand side of this formula shall vanish we derive Moreau's first integral:

In a continuously differentiable isochoric circulation-preserving motion all of whose boundaries are stationary, if $w = 0$ *and* $v = \text{const.}$ *upon any finite boundary, while in any portion extending to infinity the conditions* $vr \to 0, r^3|(\mathbf{w}\mathbf{r} \times \mathbf{v} - \mathbf{v}\mathbf{r} \times \mathbf{w}), | \to 0$ *be satisfied, then*

$$\int_{\mathfrak{B}} \mathbf{r} \times \mathbf{w} d\mathfrak{B} = \text{const.} \tag{8.9}$$

Note that if the first limit condition $vr \to 0$ be satisfied, then for the second to be satisfied also it is sufficient that $r^2 w$ be bounded.

By combining the three vectorial identities

$$\operatorname{div}(r^2 \mathbf{w}\mathbf{v}) = 2v\mathbf{r} \cdot \mathbf{w} + r^2 \mathbf{w} \cdot \operatorname{grad} \mathbf{v}, \tag{8.10}$$

$$\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) = v\mathbf{r} \cdot \mathbf{w} - w\mathbf{r} \cdot \mathbf{v}, \tag{8.11}$$

$$\operatorname{curl}(r^2 \mathbf{a}') = r^2 \operatorname{curl} \mathbf{a}' + 2\mathbf{r} \times \mathbf{a}', \tag{8.12}$$

for any motion we have

$$\begin{aligned} &2\mathbf{r} \cdot v\mathbf{w} + r^2[\operatorname{curl} \mathbf{a}' + \mathbf{w} \cdot \operatorname{grad} \mathbf{v}] \\ &\quad = -2\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) - 2\mathbf{r} \times \mathbf{a}' + \operatorname{curl}(r^2 \mathbf{a}') + \operatorname{div}(r^2 \mathbf{w}\mathbf{v}). \end{aligned} \tag{8.13}$$

By Euler's transformation (7.1), the Lagrange-Beltrami equation (7.3), and Green's transformation we thus obtain

$$\begin{aligned} -\frac{D}{Dt}\int_{\mathfrak{B}} r^2 \mathbf{w} d\mathfrak{B} &= -\int_{\mathfrak{B}} \left[2\mathbf{r} \cdot v\mathbf{w} + r^2\left(\frac{D\mathbf{w}}{Dt} + \mathbf{w} \operatorname{div} \mathbf{v}\right)\right] d\mathfrak{B}, \\ &= -\int_{\mathfrak{B}} [2\mathbf{r} \cdot v\mathbf{w} + r^2(\operatorname{curl} \mathbf{a}' + \mathbf{w} \cdot \operatorname{grad} \mathbf{v})] d\mathfrak{B}, \\ &= \int_{\mathfrak{B}} [2\mathbf{r} \times (\mathbf{v} \times \mathbf{w}) + 2\mathbf{r} \times \mathbf{a}'] d\mathfrak{B} \\ &\quad - \oint_{\mathfrak{S}} [d\mathfrak{S} \times r^2 \mathbf{a}' + d\mathfrak{S} \cdot r^2 \mathbf{w}\mathbf{v}]. \end{aligned} \tag{8.14}$$

For an isochoric motion we may express the first term on the right as a surface integral by (6.7):

$$\begin{aligned} -\frac{D}{Dt}\int_{\mathfrak{B}} r^2 \mathbf{w} d\mathfrak{B} = \int_{\mathfrak{B}} 2\mathbf{r} \times \mathbf{a}' d\mathfrak{B} - \int_{\mathfrak{S}} &[d\mathfrak{S} \times (r\mathbf{v}^2 + r^2 \mathbf{a}') \\ &+ d\mathfrak{S} \cdot (2v\mathbf{r} \times \mathbf{v} + r^2 \mathbf{w}\mathbf{v})], \end{aligned} \tag{8.15}$$

hence by the transport theorem (7.2) of Reynolds obtaining finally

$$\begin{aligned} -\frac{\partial}{\partial t}\int_{\mathfrak{B}} r^2 \mathbf{w} d\mathfrak{B} = \int_{\mathfrak{B}} 2\mathbf{r} \times \mathbf{a}' d\mathfrak{B} - \oint_{\mathfrak{S}} &[d\mathfrak{S} \times (r\mathbf{v}^2 + r^2 \mathbf{a}') \\ &+ d\mathfrak{S} \cdot (-r^2[\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v}] + 2v\mathbf{r} \times \mathbf{v})]. \end{aligned} \tag{8.16}$$

Thus we have Moreau's second integral:

In a continuously differentiable isochoric circulation-preserving motion all of whose boundaries are stationary, if $w = 0$ and $v = \text{const.}$ upon any finite boundary, while in any portion extending to infinity the conditions $r^3 v^3 \to 0$, $r^4 |(\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})_r| \to 0$ be satisfied, then

$$\int_{\mathfrak{V}} r^2 \mathbf{w} d\mathfrak{V} = \text{const.} \tag{8.17}$$

Note that if the first limit condition $r^1 v^2 \to 0$ be satisfied, then for the second to be satisfied also it is sufficient that $r^1 w^2$ be bounded. In any case the conditions sufficient for the validity of Moreau's second integral ensure the simultaneous validity of Moreau's first integral.

The kinematical conditions sufficient for the validity of Moreau's integrals are likely to be compatible with the equations of fluid dynamics (if at all) only in the case of a motion of an inviscid incompressible fluid subject to conservative extraneous force and filling all of space.

If all finite boundaries be stationary and if by some chance the non-conservative acceleration $\mathbf{a}'$ and the velocity $\mathbf{v}$ vanish upon them, then Moreau notes that (8.8) and (8.16) respectively reduce to

$$\frac{\partial}{\partial t} \int_{\mathfrak{V}} \mathbf{r} \times \mathbf{w} d\mathfrak{V} = 2 \int_{\mathfrak{V}} \mathbf{a}' d\mathfrak{V}, \tag{8.18}$$

$$-\frac{\partial}{\partial t} \int_{\mathfrak{V}} r^3 \mathbf{w} d\mathfrak{V} = 2 \int_{\mathfrak{V}} \mathbf{r} \times \mathbf{a}' d\mathfrak{V}. \tag{8.19}$$

9. Kelvin's transformation: the general integrals of A. Föppl, Jaffé, and Berker. The integrals obtained by the foregoing analyses are few, and their validity is limited. Some integrals for wider classes of motions may be found by means of a simple transformation apparently first used by Kelvin [22]: Let $\mathbf{c}$ be any solenoidal vector (div $\mathbf{c} = 0$), and let b stand for either a vector or a scalar: then

$$\text{div } \mathbf{c}b = \mathbf{c} \cdot \text{grad } b, \tag{9.1}$$

and hence

$$\int_{\mathfrak{V}} \mathbf{c} \cdot \text{grad } b d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{c}b. \tag{9.2}$$

An immediate three-dimensional generalization of Poincaré's "mass" is the *total vorticity*

$$\mathfrak{W} = \int_{\mathfrak{V}} \mathbf{w} d\mathfrak{V}, \tag{9.3}$$

introduced by A. Föppl[10] [4]. Putting $\mathbf{c} = \mathbf{w}$, $b = \mathbf{r}$ in (9.2) we obtain

$$\mathfrak{W} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{w}\mathbf{r}, \tag{9.4}$$

and hence Föppl's integral:

[10]The formula (9.4) was apparently known to Föppl, though he did not state it explicitly. It is equation (7) of [17].

In any continuously differentiable motion such that

a) *upon all finite boundaries the normal component of vorticity vanish, and*

b) *in any portion of the material extending to infinity the condition* $r^3 w_r \to 0$ *be satisfied, then the total vorticity vanishes*:

$$\mathfrak{W} = 0. \tag{9.5}$$

As special cases we notice that the result holds for the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping.

It was shown by Jaffé [7] that Poincaré's theorem of the conservation of "mass" can be extended to three-dimensional motions of viscous incompressible fluids filling all space and vanishing suitably at infinity; that is, the total vorticity of such motions is constant in time. In an earlier paper [23] I showed that Jaffé's theorem can be expressed in a purely kinematical form, valid for any continuous medium; a proof will be given incidentally in §12 of this memoir.

Three other special cases of Kelvin's transformation have been published recently by Berker[11] [2, 30], as follows.

First, write $\mathbf{f} = \operatorname{grad} b$ in (9.2):

$$\int_{\mathfrak{B}} \mathbf{c} \cdot \mathbf{f} \, d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{c} b. \tag{9.6}$$

Hence we conclude a generalization of the theorem III of Berker:

Let $\mathbf{c}$ *be a continuously differentiable solenoidal field whose normal component upon any finite boundary of* $\mathfrak{B}$ *is zero, and let* $\mathbf{f} = \operatorname{grad} b$ *be any continuous laminar field (vector or dyadic), and suppose further that in any portion of* $\mathfrak{B}$ *extending to infinity the condition* $r^3 b c_r \to 0$ *be satisfied; then*

$$\int_{\mathfrak{B}} \mathbf{c} \cdot \mathbf{f} \, d\mathfrak{V} = 0. \tag{9.7}$$

Second, since $\operatorname{div} \mathbf{w} = 0$, we may put $\mathbf{w} = \mathbf{c}$ in (9.6) and obtain

$$\int_{\mathfrak{B}} \mathbf{w} \cdot \mathbf{f} \, d\mathfrak{V} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{w} b, \tag{9.8}$$

hence concluding a generalization of Berker's theorem II:

Let $\mathbf{w}$ *be any continuous vorticity field whose normal component upon any finite boundary of* $\mathfrak{B}$ *is zero, and let* $\mathbf{f} = \operatorname{grad} \mathbf{b}$ *be any continuous laminar field (vector or dyadic), and suppose further that in any portion of* $\mathfrak{B}$ *extending to infinity the condition* $r^3 b w_r \to 0$ *be satisfied, then*

[11] A fourth result of Berker has been noted in §6.

$$\int_{\mathfrak{B}} \mathbf{w} \cdot \mathbf{f} d\mathfrak{B} = 0. \tag{9.9}$$

Notice the conditions of this theorem are satisfied by the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping (Berker's case). Putting $\mathbf{f} = \mathbf{I} = \text{grad } \mathbf{r}$, where $\mathbf{I}$ is the unit dyadic, we obtain (9.5) as a special case. Putting $\mathbf{f} = \mathbf{r} = \text{grad } r^2/2$, we obtain

$$\int_{\mathfrak{B}} \mathbf{r} \cdot \mathbf{w} d\mathfrak{B} = 0. \tag{9.10}$$

Third, let $\mathbf{f}$ be a vector field such that $\text{div } \mathbf{f} = 0$, $\nabla^2 \mathbf{f} = 0$; equivalently, $\text{curl curl } \mathbf{f} = 0$, or $\text{curl } \mathbf{f} = \text{grad } b$, say. Let $\mathbf{c} = \mathbf{v}$, and suppose $\text{div } \mathbf{v} = 0$. Then, Kelvin's transformation (9.2) becomes

$$\int_{\mathfrak{B}} \text{curl } \mathbf{f} \cdot \mathbf{v} d\mathfrak{B} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot \mathbf{v} b. \tag{9.11}$$

Now

$$\text{curl } \mathbf{f} \cdot \mathbf{v} = \text{div } (\mathbf{f} \times \mathbf{v}) + \mathbf{f} \cdot \mathbf{w}; \tag{9.12}$$

hence (9.11) becomes

$$\int_{\mathfrak{B}} \mathbf{f} \cdot \mathbf{w} d\mathfrak{B} = \oint_{\mathfrak{S}} d\mathfrak{S} \cdot [\mathbf{v} b + \mathbf{v} \times \mathbf{f}], \tag{9.13}$$

whence follows a generalization of Berker's theorem I:

In any continuously differentiable isochoric motion such that upon any finite boundaries the material adheres without slipping, if $\mathbf{f}$ *be any twice continuously differentiable field such that* $\text{curl curl } \mathbf{f} = 0$, *and hence* $\text{curl } \mathbf{f} = \text{grad } b$, *and if further in any portion of the motion extending to infinity the conditions* $r^2 v, b \to 0$, $r^2 vf \to 0$ *be satisfied, then*

$$\int_{\mathfrak{B}} \mathbf{f} \cdot \mathbf{w} d\mathfrak{B} = 0. \tag{9.14}$$

It is interesting to notice that the form of both the integrals (9.9) and (9.14) is the same, although the meaning is different. The former is not restricted to isochoric motions, as is the latter, but in the former the field $\mathbf{f}$ must satisfy $\text{curl } \mathbf{f} = 0$, while in the latter it need satisfy only $\text{curl curl } \mathbf{f} = 0$.

Part III. The General Conservation Theorems

In the following sections we shall show that a slight generalization of Kelvin's transformation enables the construction of an infinite sequence of moments of vorticity which themselves can be expressed as surface integrals, and whose derivatives may be so expressed, and thus we shall obtain two sequences of possible integrals, which include those discussed in §9 as the simplest special cases.

10. Symmetrical moments of vorticity. For the polyadic nth power of a vector $\mathbf{b}$ let us introduce the notation $\mathbf{b}^n$:

$$\mathbf{b}^0 = 1, \quad \mathbf{b}^1 = \mathbf{b}, \quad \mathbf{b}^2 = \mathbf{bb}, \quad \mathbf{b}^3 = \mathbf{bbb}, \ldots \tag{10.1}$$

(In this paper the notation $\mathbf{r}^2$ will not be used in its usual sense of $\mathbf{r}\cdot\mathbf{r}$, which we shall denote instead by r^2.) Then let the notation $\{\mathbf{b}^n\mathbf{c}\}$ stand for the completely symmetrical polyadic

$$\{\mathbf{b}^n\mathbf{c}\} = \mathbf{b}^n\mathbf{c} + \mathbf{b}^{n-1}\mathbf{cb} + \mathbf{b}^{n-2}\mathbf{cb}^2 + \ldots + \mathbf{cb}^n; \tag{10.2}$$

thus *e.g.*

$$\{\mathbf{b}^0\mathbf{c}\} = \mathbf{c}, \quad \{\mathbf{b}^1\mathbf{c}\} = \mathbf{bc} + \mathbf{cb}, \quad \{\mathbf{b}^2\mathbf{c}\} = \mathbf{bbc} + \mathbf{bcb} + \mathbf{cbb}. \tag{10.3}$$

We then define the *nth symmetrical moment of vorticity* $\mathfrak{M}_n$ by

$$\mathfrak{M}_n = \int_{\mathfrak{V}} \{\mathbf{r}^n\mathbf{w}\}\, d\mathfrak{V}, \tag{10.4}$$

where $\mathfrak{V}$ is an arbitrary finite domain, within whose interior $\mathbf{w}$ is assumed to be continuously differentiable. Comparison of the first three moments

$$\begin{aligned} \mathfrak{M}_0 &= \int_{\mathfrak{V}} \mathbf{w}\, d\mathfrak{V}, \\ \mathfrak{M}_1 &= \int_{\mathfrak{V}} (\mathbf{rw} + \mathbf{wr})\, d\mathfrak{V}, \\ \mathfrak{M}_2 &= \int_{\mathfrak{V}} (\mathbf{rrw} + \mathbf{rwr} + \mathbf{wrr})\, d\mathfrak{V}, \end{aligned} \tag{10.5}$$

with (9.3), (2.3), (9.10) and (2.4) shows that $\mathfrak{M}_0$ is Föppl's total vorticity $\mathfrak{M}$, that either the z-row or the z-column of $\mathfrak{M}_1$ taken over a cylinder of unit height in the case of a plane motion is a vector whose components are Poincaré's $\mathfrak{W}x_w$, $\mathfrak{W}y_w$, and $\mathfrak{W}$, that the scalar of $\mathfrak{M}_1$ is one of Berker's integrals, and that Poincaré's $\mathfrak{I}$ is the sum of two of the components of $\mathfrak{M}_2$ taken over a cylinder of unit height in the case of a plane motion.

11. The first general conservation theorem. Since $\operatorname{div}\mathbf{w} = 0$, it is easy to see that

$$\operatorname{div}[\mathbf{wr}^{n+1}] = \{\mathbf{r}^n\mathbf{w}\} \tag{11.1}$$

Putting this result into (10.4), by Green's transformation we obtain an expression for $\mathfrak{M}_n$ as a surface integral:

$$\mathfrak{M}_n = \int_{\mathfrak{S}} d\mathfrak{S}\cdot\mathbf{wr}^{n+1}. \tag{11.2}$$

Hence *all the moments $\mathfrak{M}_n$ are independent of the motion at interior points, being completely determined by the normal component of vorticity upon the bounding*

surfaces. This purely kinematical result illustrates the predominant effect of boundaries upon vorticity. It may be regarded as indicating that the familiar hydrodynamical theorem that vorticity cannot be generated in the interior of a homogeneous viscous liquid subject to conservative extraneous force, but must be diffused inward from the boundaries, continues to hold for arbitrary continuous media, provided it be expressed in terms of the average rather than the local vorticity. In particular, we conclude the following first general conservation theorem:

If upon any finite boundary of a continuously differentiable motion the normal component w_n of the vorticity be zero, while in any portion extending to infinity the radial component w_r satisfy the condition

$$r^{n+2}w_r \to 0, \tag{11.3}$$

then the first $n+1$ symmetrical moments of vorticity vanish:

$$\mathfrak{W}_0 = \mathfrak{W}_1 = \ldots = \mathfrak{W}_n = 0. \tag{11.4}$$

As special cases we may notice that the result holds *for all n* for the material within a closed vortex-tube of any continuously differentiable motion; for the entire material of any continuously differentiable motion in a bounded domain upon whose boundaries $w = 0$; and, by (7.5), for any such motion within finite stationary boundaries to which the material adheres without slipping.

12. The second general conservation theorem. While the first conservation theorem yields an infinite number of integrals for many motions, including even so general a case as any continuously differentiable motion of a non-homogeneous viscous compressible fluid of variable viscosity in a finite domain, for motions extending to infinity the limit condition (11.3) may be satisfied only for the first few values of n. It is possible, however, that for some larger values of n the symmetrical moments of vorticity while not zero may nevertheless remain constant in time, as the following analysis shows.

First, let the boundary surface $\mathfrak{S}$ be a material surface. Then from (11.2) we have

$$\frac{D\mathfrak{W}_n}{Dt} = \oint_{\mathfrak{S}} \frac{D}{Dt}\,(d\mathfrak{S}\cdot\mathbf{w}r^{n+1}), \tag{12.1}$$

$$= \oint_{\mathfrak{S}}\left[d\mathfrak{S}\cdot\left(\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} + \frac{D\mathbf{w}}{Dt}\,\mathbf{r}^{n+1}\right) + \frac{Dd\mathfrak{S}}{Dt}\cdot\mathbf{w}r^{n+1}\right].$$

But if we employ the Lagrange-Beltrami equation (7.3) and the formula

$$\frac{Dd\mathfrak{S}}{Dt} = -\operatorname{grad}\mathbf{v}\cdot d\mathfrak{S} + \operatorname{div}\mathbf{v}\,d\mathfrak{S}, \tag{12.2}$$

of Lamb [11], by a happy circumstance the convective portions of these two expressions cancel each other and (12.1) reduces simply to

$$\frac{D\mathfrak{M}_n}{Dt} = \oint_{\mathcal{S}} d\mathfrak{S} \cdot \mathbf{w}\{\mathbf{r}^n\mathbf{v}\} + \oint_{\mathcal{S}} d\mathfrak{S} \cdot \operatorname{curl} \mathbf{a}'\mathbf{r}^{n+1}. \tag{12.3}$$

The former integral is the convective rate of change of $\mathfrak{M}_n$, the latter is the diffuse rate of change[12]. To obtain the Eulerian derivative of $\mathfrak{M}_n$ for a fixed volume, we need only employ the Reynolds transport theorem (7.2), thus obtaining

$$\frac{\partial\mathfrak{M}_n}{\partial t} = \oint_{\mathcal{S}} d\mathfrak{S} \cdot [\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} - \mathbf{v}\{\mathbf{r}^n\mathbf{w}\}] + \oint_{\mathcal{S}} d\mathfrak{S} \cdot \operatorname{curl} \mathbf{a}'\mathbf{r}^{n+1}. \tag{12.4}$$

Since we have identically

$$\begin{gathered} d\mathbf{b} \cdot \operatorname{curl} (\mathbf{r}^{n+1}\mathbf{c}) = -\{\mathbf{r}^n(d\mathbf{b} \times \mathbf{c})\} + d\mathbf{b} \cdot \operatorname{curl} \mathbf{c}\, \mathbf{r}^{n+1}, \\ \oint_{\mathcal{S}} d\mathfrak{S} \cdot \operatorname{curl} \Sigma = \int_{\mathfrak{V}} \operatorname{div} \operatorname{curl} \Sigma\, d\mathfrak{V} = 0, \end{gathered} \tag{12.5}$$

we may put the diffusive term of (12.4) into another form, obtaining finally the not inelegant relation

$$\frac{\partial\mathfrak{M}_n}{\partial t} = \oint_{\mathcal{S}} d\mathfrak{S} \cdot [\mathbf{w}\{\mathbf{r}^n\mathbf{v}\} - \mathbf{v}\{\mathbf{r}^n\mathbf{w}\}] + \int_{\mathcal{S}} \{\mathbf{r}^n(d\mathfrak{S} \times \mathbf{a}')\}. \tag{12.6}$$

The special case $n = 0$, generalizing a formula of Jaffé [7], I gave in an earlier paper [23]. From (12.6) follows the second general conservation theorem:

If all finite boundaries of a continuously differentiable motion be stationary, and if upon them the normal component w_n of vorticity be zero, and either or both of the conditions

$$a_t = 0, \quad a'_t = 0 \tag{12.7}$$

be satisfied, while in any portion of the material extending to infinity the condition

$$r^{n+2}(vw_r - wv_r - a'_t) \to 0 \tag{12.8}$$

be satisfied, then the first $n + 1$ symmetrical moments of vorticity are constant in time:

$$\mathfrak{M}_0 = \text{const.}, \quad \mathfrak{M}_1 = \text{const.}, \ldots, \mathfrak{M}_n = \text{const.} \tag{12.9}$$

Notice that in the case when all finite boundaries are stationary and the material adheres without slipping, we have $a_t = 0$ as well as (7.5), and hence the only condition of the theorem which remains to be considered is the limit condition (12.8) at infinity. Jaffé's theorem is included in the case $n = 0$.

13. Conclusion. To the best of my knowledge the foregoing theorems furnish the first instances of an infinite set of possible integrals for a class of three-dimensional motions of a continuum, even for a continuum of a special

[12] For the terms "convection" and "diffusion", see [24].

type. The analysis is purely kinematical, and thus the results as stated are independent of whatever dynamical characteristics the medium may possess. The dynamical response of a continuum is generally specified by giving a formula for the acceleration, and thus will determine for what values of n the condition of the theorems will be satisfied. In general it may be expected that a motion vanishing at infinity[13] will satisfy the conditions of the first theorem for a small value of n, and those of the second theorem for a somewhat larger n, so that the first few moments of vorticity will vanish and the next one or two will remain constant.

[13]Since a plane motion does not vanish at infinity, no plane motion can satisfy the conditions of our two general theorems. If we attempt to apply them to the region confined between the planes $z = 0$ and $z = 1$ in a plane motion, we find that the normal component of vorticity does not vanish upon these planes, and hence again the theorems are inapplicable. It is this fact which justifies the statement in §1 that three-dimensional motions are simpler in their average properties than are plane motions.

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AUTOMETRIZED BOOLEAN ALGEBRAS I: FUNDAMENTAL DISTANCE-THEORETIC PROPERTIES OF B

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1. Introduction. There have been several brief studies made [3, 4, 7, 8, 9, 11] of systems in which a "distance function" is defined on the set of pairs of elements of some abstract set to another abstract set. Frequently both of the sets involved are given algebraic structures. One of the more novel of these systems is the naturally metrized group [3, 7] originated by Karl Menger in 1931. This system is analogous to the Euclidean line in that it assigns to each pair, a, b of elements of an additively written Abelian group the "absolute value", $(a\text{-}b, b\text{-}a) = (b\text{-}a, a\text{-}b)$, of the "difference" of the elements as "distance". As might be anticipated (since this is precisely the distance function used on the Euclidean line) many of the results valid for the distance function on the Euclidean line are also valid for this "distance function"

In this paper a system will be defined which is also somewhat similar to the Euclidean line in that the "distances" of elements are themselves elements of the same set. A few of the usual notions of metric geometry will then be examined in terms of these "distances". It should be noted that the definitions of metric concepts in this system are analogous to the corresponding definitions for metric spaces; in particular, there is a striking resemblance between the list of properties given in Theorem 1.1 and the usual list of assumptions about the metric of a metric space.

DEFINITION. Let B be a Boolean algebra with meet, join, complement, and inclusion (in the wide sense) denoted by ab, $a+b$, a', and $a \subset b$ respectively. (All these notions are discussed in [1]. The notions of metric geometry not defined in this paper are discussed in L. M. Blumenthal's *Distance Geometries*, University of Missouri Studies, vol. XIII (1938).)

Define "distance" of a and b by $d(a, b) = ab' + a'b$. Throughout the paper subsets of B will be denoted by capital English letters and all small English letters will denote elements of B. The name, *autometrized Boolean algebra*, seems appropriate for the system thus defined.

Remark (i). The "distance function" defined in B, $d(a, b) = ab' + a'b$, is also the element corresponding to the ring sum of a and b under the one-to-one correspondence of Boolean algebras and Boolean rings with units [1] so all the results of this paper might, if desired, be re-phrased as results concerning the ring sum in a Boolean ring with a unit.

Received September 29, 1949. This paper was presented to the American Mathematical Society November 25, 1949, and constitutes Part III of a dissertation being written for the Ph.D. degree at the University of Missouri under the direction of L. M. Blumenthal.

Remark (ii). Since $d(a, b) = ab' + a'b = (a + b)(a' + b')$,
$d'(a, b) = ab + a'b' = (a' + b)(a + b')$.

Remark (iii). The "distance function" is invariant with respect to complementation; that is, $d(a, b) = d(a', b')$.

Remark (iv). The first element[1] of B acts as an "origin" with respect to the "distance function"; that is, $d(0, a) = a$.

THEOREM 1.1. *The "distance function" has the following properties:*

1. *Symmetry:* $d(a, b) = d(b, a)$.
2. *Vanishing:* $d(a, b) = 0$ *if and only if* $a = b$.
3. *Triangle inequality:* $d(a, c) \subset d(a, b) + d(b, c)$.

Proof. Part 1 is obvious and it is trivial to verify that $d(a, a) = 0$ for part 2. Suppose now that $d(a, b) = 0$. Then $ab' = a'b = 0$. Taking complements in $ab' = 0$, one finds that $a' + b = 1$. Hence $a' = b'$ and $a = b$.

To prove part 3 take

$$(d(a, b) + d(b, c))\, d(a, c) = (ab' + a'b + bc' + b'c)(ac' + a'c)$$
$$= a'bc + a'b'c + ab'c' + abc' = a'c(b + b') + ac'(b' + b) = a'c + ac'$$
$$= d(a, c).$$

2. Betweenness.

DEFINITION. An element b *is between* a *and* c provided

$$d(a, c) = d(a, b) + d(b, c).$$

Pitcher and Smiley [10] have laid the foundations for a theory of betweenness in general lattices based on the condition of Glivenko [5, 6]; that is, b *is* G-*between* a *and* c provided

(G) $$ab + bc = b = (a + b)(b + c).$$

We shall show that these two varieties of betweenness are equivalent in B.

LEMMA 2.1. *The element* b *is* G-*between* a *and* c *if and only if* b' *is* G-*between* a' *and* c'.

Proof. This result follows from complementation in condition (G).

THEOREM 2.1. *The element* b *is between* a *and* c *if and only if* b *is* G-*between* a *and* c.

Proof. Suppose first that b is G-between a and c. Then

$$ab + bc = b = (a + b)(b + c)$$

and by Lemma 2.1, $a'b' + b'c' = b' = (a' + b')(b' + c')$.

[1]The symbols 0 and 1 denote the first and last elements of B.

Now $d(a, b) + d(b, c) = a'b + ab' + b'c + bc' = b(a' + c') + b'(a + c)$
$= (ab + bc)(a' + c') + (a'b' + b'c')(a + c)$
$= a'bc + abc' + ab'c' + a'b'c$
$= a'c(b + b') + ac'(b + b') = a'c + ac' = d(a,c).$

Hence b is between a and c.

Suppose now that b is between a and c. Then

$$a'b + ab' + b'c + bc' = a'c + ac'.$$

Taking meets in this equation with a, a', b, and b' respectively it is found that

(1) $ab' + ab'c + abc' = ac',$
(2) $a'b + a'b'c + a'bc' = a'c,$
(3) $a'b + bc' = a'bc + abc',$
(4) $ab' + b'c = a'b'c + ab'c'.$

Taking meets in 1 with b', c, and c' respectively; in 2 with b, c, and c' respectively; in 3 with c'; and in 4 with c one obtains:

(5) $ab' + ab'c = ab'c',$
(6) $ab'c + ab'c = 0,$
(7) $ab'c' + abc' = ac',$
(8) $a'b + a'bc' = a'bc,$
(9) $a'bc + a'b'c = a'c,$
(10) $a'bc' + a'bc' = 0,$
(11) $a'bc' + bc' = abc',$
(12) $ab'c + b'c = a'b'c.$

From (6) and (10) one obtains:

(13) $ab'c = 0,$
(14) $a'bc' = 0.$

From (13) and (5), (14) and (8), (14) and (11), and (13) and (12), respectively, one finds that:

(15) $ab' = ab'c'$ or $ab' \subset c',$
(16) $a'b = a'bc$ or $a'b \subset c.$
(17) $bc' = abc'$ or $bc' \subset a.$
(18) $b'c = a'b'c$ or $b'c \subset a'.$

Combining (7), (15), and (17), and then combining (9), (16), and (18) one obtains:

(19) $ab' + bc' = ac',$
(20) $a'b + b'c = a'c.$

(Note that we have in effect "factored" $d(a, c) = d(a, b) + d(b, c)$ into equations (19) and (20)).

Complementation in (19) and (20) yields:

$$(a' + b)(b' + c) = a' + c, \tag{21}$$
$$(a + b')(b + c') = a + c'. \tag{22}$$

Taking meets between (21) and (22) yields

$$abc + a'b'c' = ac + a'c'. \tag{23}$$

Taking meets in (23) with ac and $a'c'$ respectively we find that:

$$abc = ac \text{ or } ac \subset b, \tag{24}$$
$$a'b'c' = a'c' \text{ or } a'c' \subset b'. \tag{25}$$

From (24), $b = b + ac = (a + b)(b + c)$. Hence

$$b = (a + b)(b + c). \tag{26}$$

Complementation in (25) yields $a + b + c = a + c$ or $b \subset a + c$ so that $b = b(a + c) = ab + bc$. Hence

$$b = ab + bc. \tag{27}$$

Then combining (26) and (27) we find

$$ab + bc = b = (a + b)(b + c). \tag{G}$$

Hence b is G-between a and c.

Remark (i). All of the properties of G-betweenness in general lattices demonstrated by Pitcher and Smiley [10] are valid for betweenness in B.

Remark (ii). If B is a *metric lattice* [1] then betweenness in B is equivalent to *metric betweenness* (in the wide sense) since Glivenko [5, 6] showed that metric betweenness (in the wide sense) and G-betweenness are equivalent in any metric lattice.

Remark (iii). Betweenness in B is equivalent to each of the following:

$$ab + bc = b = (a + b)(b + c), \tag{G}$$
$$ab + bc = b = b + ac, \tag{G*}$$
$$b(a + c) = b = (a + b)(b + c). \tag{G**}$$

Proof. Equivalence of betweenness and (G) was demonstrated in Theorem 2.1. It is obvious that (G), (G*), and (G**) are pairwise equivalent in any *distributive* lattice (in fact, L. M. Blumenthal and the writer have shown that these conditions are pairwise equivalent in any *modular* lattice [2]).

3. The group of motions of B. Superposability properties.

DEFINITIONS. Let E and F be subsets of B. E *is congruent to* F, written $E \approx F$, provided there is a single-valued mapping $f: E \to F$ such that for $a, b \in E$, $d(f(a), f(b)) = d(a, b)$. It is easily seen that the mapping f is then biuniform and that F is congruent to E by the mapping $f^{-1}: F \to E$. The mapping f (or f^{-1}) is then called a *congruence* between E and F.

A *motion* of B is a congruence of B with itself. It is clear that these motions form a group of transformations. This group is called *transitive* (simply transitive) provided there is a motion of B sending any given element of B into any other given element. Two subsets, E and F, are called *superposable*, written $E \simeq F$, provided there is a motion of B mapping one onto the other. Clearly $E \approx F$ if $E \simeq F$.

B is said to have the *property of n-element superposability* provided that if E and F are each sets of n elements and $E \approx F$ then $E \simeq F$.

B is said to have the *property of free mobility* provided *any* congruence between *any* two subsets of B may be extended to a motion of B; that is, provided there is a motion of B which induces the *original* congruence between the two given sets.

THEOREM 3.1. *B has the property of two-element superposability.*

Proof. Let $a, b \approx m,n$. Define a mapping of B,

$$f(x) = mn(xab + x'a'b') + m'n'(x'ab + xa'b') + mn'(xab' + x'a'b) + m'n(xa'b + x'ab').$$

This mapping is single valued and sends a into m and b into n.
For example,

$$\begin{aligned} f(a) &= mn(ab + a'b') + mn'(ab' + a'b) \\ &= mn(mn + m'n') + mn'(mn' + m'n) \\ &= mn + mn' = m. \end{aligned}$$

By laborious expansion and collection of meets one may verify that

$$\begin{aligned} &d(f(x), f(y)) \\ &\quad = f(x)\, f'(y) + f'(x)\, f(y) \\ &\quad = (x'y + xy')\,((ab + a'b')\,(mn + m'n') + (a'b + ab')\,(m'n + mn')) \\ &\quad = (x'y + xy')\,(ab + a'b' + a'b + ab') = x'y + xy' = d(x,y) \end{aligned}$$

so that $f(x)$ is a congruence of B with itself.

Remark. The group of motions of B is transitive as a direct consequence of Theorem 3.1.

THEOREM 3.2. *B has the property of free mobility.*

Proof. Let E and F be any two congruent subsets of B and let $g\colon E \to F$ be this congruence. If E contains less than three distinct elements the proof reduces to an application of Theorem 3.1. Otherwise, select $a, b \in E$, $a \neq b$. Then $a,b \approx g(a)$, $g(b)$ and $f(x)$ as defined in the proof of Theorem 3.1 sends a into $g(a)$ and b into $g(b)$. But $f(x)$ is a motion of B so it remains only to verify that f induces g on E. Let then $c \in E$. Since g is a congruence, $a, b, c \approx g(a), g(b), g(c)$.
From the six relations

$$\begin{aligned} a'b + ab' &= g'(a)g(b) + g(a)g'(b), & ab + a'b' &= g(a)g(b) + g'(a)g'(b), \\ a'c + ac' &= g'(a)g(c) + g(a)g'(c), & ac + a'c' &= g(a)g(c) + g'(a)g'(c), \\ b'c + bc' &= g'(b)g(c) + g(b)g'(c), & bc + b'c' &= g(b)g(c) + g'(b)g'(c) \end{aligned}$$

resulting from this congruence, one obtains

$$\begin{aligned} abc + a'b'c' &= g(a)g(b)g(c) + g'(a)g'(b)g'(c), \\ abc' + a'b'c &= g(a)g(b)g'(c) + g'(a)g'(b)g(c), \\ ab'c + a'bc' &= g(a)g'(b)g(c) + g'(a)g(b)g'(c), \\ a'bc + ab'c' &= g'(a)g(b)g(c) + g(a)g'(b)g'(c). \end{aligned}$$

Then

$$\begin{aligned} f(c) &= g(a)g(b)(g(a)g(b)g(c) + g'(a)g'(b)g'(c)) \\ &\quad + g'(a)g'(b)(g(a)g(b)g'(c) + g'(a)g'(b)g(c)) \\ &\quad + g'(a)g(b)(g'(a)g(b)g(c) + g(a)g'(b)g'(c)) \\ &\quad + g(a)g'(b)(g(a)g'(b)g(c) + g'(a)g(b)g'(c)) \\ &= g(a)g(b)g(c) + g'(a)g'(b)g(c) + g(a)g'(b)g(c) + g'(a)g(b)g(c) \\ &= g(c)\,(g(a)g(b) + g'(a)g'(b) + g(a)g'(b) + g'(a)g(b)) \\ &= g(c). \end{aligned}$$

Hence f induces g on E, and Theorem 3.2 is proved.

4. Other properties of "distance" in B.

THEOREM 4.1. *Let $a, b \in B$ and $d(a, b) = c$. Then $d(a, c) = b$ and $d(b, c) = a$.*

Proof. $d(a, b) = a'b + ab' = c$ so that $a'b = a'c$. $d'(a, b) = c' = ab + a'b'$ so that $ab = ac'$. Hence $b = ab + a'b = ac' + a'c = d(a, c)$. Similarly one shows that $a = d(b, c)$.

THEOREM 4.2. *Let $c \in B$ and $a \in B$. There is exactly one element b of B so that $d(a, b) = c$. This is sometimes stated by saying that any given element forms a metric basis for B.*

Proof. Let $b = d(a, c)$. Then by Theorem 4.1, $c = d(a, b)$. Hence there is at least one such element b. Suppose that $d(a, x) = c$. Then by Theorem 4.1, $x = d(a, c) = b$ so that the element b is unique.

Remark. Isosceles and equilateral triples, common structures in metric spaces, are shown by Theorem 4.2 to be absent from B.

5. The "metric characterization" of B.

DEFINITION. Let Σ be an abstract set, to each pair of elements α, β of which is attached an element $d(\alpha, \beta)$ of B such that $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$, and $d(\alpha, \beta) = d(\beta, \alpha)$. Such a set will be called a B-*metrized space.*

B is said to have *congruence order* n with respect to the class of B-metrized spaces provided any B-metrized space is congruently contained (congruent to a subset of) in B whenever each n elements of the space are congruently contained in B. The smallest integer n for which B has congruence order n is called the *best congruence order* of B. In the following theorem small greek letters will denote elements of a B-metrized space.

THEOREM 5.1. *Let B be any autometrized Boolean algebra. The best congruence order of B with respect to the class of* B-*metrized spaces is three.*

Proof. We shall first show that B has congruence order three and then that this is the best congruence order of B.

Let Σ be any B-metrized space with more than two distinct elements (otherwise the proof is trivial), and suppose that each three elements of Σ are congruent with three elements of B. Select $\alpha, \beta \in \Sigma$, $\alpha \neq \beta$. Then $\alpha, \beta \approx a, b \in B$. Let $\xi \in \Sigma$. Now $\alpha, \beta, \xi \approx a_1, b_1, x_1 \in B$. But $a_1, b_1 \simeq a, b$ by Theorem 3.1. Let x be the image of x_1 under this motion. Then $\alpha, \beta, \xi \approx a, b, x$ and we have established a single-valued mapping $x = x(\xi)$ of Σ into B. It remains to show that "distances" are preserved. Let $\xi, \eta \in \Sigma$ and let x, y be the elements of B corresponding to ξ, η under the previously defined mapping. Now $\alpha, \xi, \eta \approx a_2, x_2, y_2 \in B$ by hypothesis. But $a, x \approx \alpha, \xi$. Hence by Theorem 3.1, $a, x \simeq a_2, x_2$. Let y_3 be the image of y_2 under the motion sending a_2 into a and x_2 into x. Then $a, y_3 \approx a_2, y_2 \approx \alpha, \eta \approx a, y$. Hence by Theorem 4.2, $y_3 = y$ so that $a, x, y \approx a_2, x_2, y_2 \approx \alpha, \xi, \eta$ and $d(x, y) = d(\xi, \eta)$. This establishes that Σ is congruently contained in B and that B has congruence order three.

The example consisting of three elements α, β, γ with $d(\alpha, \beta) = d(\beta, \gamma) = d(\alpha, \gamma) = a$, $d(\alpha, \alpha) = d(\beta, \beta) = d(\gamma, \gamma) = 0$ where a is any element of B different from 0 shows that B cannot have congruence order less than three since any proper subset of this B-metrized space is congruently contained in B but the space itself forms an equilateral triple and so cannot be congruently contained in B which is free from equilateral triples.

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FINITE NETS, I. NUMERICAL INVARIANTS

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Introduction. A finite net N of degree k, order n, is a geometrical object of which the precise definition will be given in §1. The geometrical language of the paper proves convenient, but other terminologies are perhaps more familiar. A finite affine (or Euclidean) plane with n points on each line ($n \geqslant 2$) is simply a net of degree $n + 1$, order n (Marshall Hall [1]). A loop of order n is essentially a net of degree 3, order n (Baer [1], Bates [1]). More generally, for $3 \leqslant k \leqslant n + 1$, a set of $k - 2$ mutually orthogonal $n \times n$ latin squares may be used to define a net of degree k, order n (and conversely) by paralleling Bose's correspondence (Bose [1]) between affine planes and complete sets of orthogonal latin squares.

In the language of latin squares, the problem (explained in §1) of imbedding a net of N of degree k, order n in a net N' of degree $k + 1$, order n becomes the problem of finding an $n \times n$ latin square orthogonal to each of $k - 2$ given mutually orthogonal $n \times n$ latin squares. Similarly, adjunction of a line corresponds to the determination of a common "transversal" (in the terminology of Euler [1]) to the $k-2$ orthogonal squares. Further details of a historical nature will be found in the bibliography.

On each finite net N we define an integer $\phi(N)$, which may be regarded as an invariant in several ways. A necessary condition that a line can be adjoined to N is that $\phi(N) = 1$. (A necessary and sufficient condition is given in Theorem 1 (i).) We define a direct product $N_1 \times N_2$ of nets N_i of the same degree and study the relation between $\phi(N_1 \times N_2)$ and the $\phi(N_i)$ (Theorem 4). From these considerations we deduce the existence of nets of every order n to which no line can be adjoined (Theorem 5). Next we study the relation between the ϕ's of homomorphic nets (Theorem 6) and we conclude the paper with an explicit evaluation of ϕ for nets of degree 3 (Theorem 7).

1. Nets and the imbedding problem. Let k, n be positive integers, with $k \geqslant 3$. A (finite) *net* N of degree k, order n, is a system of undefined objects called "points" and "lines" together with an incidence relationship ("point is on line" or "line passes through point") such that: (i) N contains k (non-empty) classes of lines. (ii) Two lines a, b of N, belonging to distinct classes, have a unique common point P. (iii) Each point P of N is on exactly one line of each class. (iv) Some line of N has exactly n distinct points. It is easy to show that every line of N has exactly n distinct points, that every class of lines contains exactly n distinct lines and that N consists of n^2 distinct points, kn distinct lines. Moreover, either $n = 1$ or $n \geqslant k - 1$.

Received October 8, 1949.

If S is a subset of the points of the net N (of degree k, order n) such that each line of N contains exactly one point of S, we shall say that *S can be adjoined as a line to N*. Considering the n lines of each class, we see that S must consist of exactly n distinct points, no two collinear. If the n^2 points of N can be partitioned into n disjoint sets $S_1, \ldots, S_n$, each of which can be adjoined as a line to N, then the S_j may be regarded as constituting the n lines of an additional class. In this way N can be *imbedded* in a net N' of degree $k+1$, order n, consisting of the points and lines of N (with the same incidence relations) plus one additional class of "parallels". Conversely, if the net N of order n, degree k is a subnet of net N' of order n, degree $k+1$ (a subnet in the sense that a point and line of N are incident in N if and only if they are incident in N') then N, N' must have the same points, and one of the line-classes of N' may be regarded as consisting of n disjoint point-sets S_j, each of which can be adjoined as a line to N. The present paper will be concerned primarily with *necessary* conditions that a line may be adjoined to a net.

2. The integers represented by a net. Let N be a finite net and let f be a single-valued function from the points of N to the rational integers. We shall say that the rational integer m is *represented on N by f* if f sums to m over the points of each line of N, and *represented positively* if, in addition, f takes on only non-negative values. Again, if u is a positive integer, we shall say that m is *represented* mod u *on N by f* if f sums to m mod u on each line of N. The least positive integer represented on N will be denoted by $\phi(N)$. Clearly $\phi(N)$ is an invariant of N. Moreover, $\phi(N)$ is the (positive) greatest common divisor of the integers represented on N.

THEOREM 1. *Let N be a finite net of degree k, order n. Then*: (i) *A necessary and sufficient condition that a line can be adjoined to N is that 1 be positively represented on N.* (ii) *n is positively represented on N.* (iii) *$k-1$ is represented on N.* (iv) $\phi(N) \mid (n, k-1)$. (v) *If n is an affine plane (i.e., if $k = n+1$,) $\phi(N) = n$.* (vi) *With at most a finite number of exceptions, every positive integer divisible by $\phi(N)$ is positively represented on N.*

COROLLARY. *A necessary condition that a line can be adjoined to N is that* $\phi(N) = 1$.

Proof. (i) If S can be adjoined as a line to N, define $f(P) = 1$ or 0 according as P is or is not in S. Then 1 is positively represented on N by f. Conversely, if 1 is positively represented on N by some f, let S be the set of points P for which $f(P) \neq 0$. Then each line of N contains exactly one point P of S (and, incidentally, $f(P) = 1$.) Hence S can be adjoined to N as a line.

(ii) If $f'(P) = 1$ for every point P of N, then f' represents n positively on N.

(iii) Select an arbitrary point C of N and define h as follows: $h(C) = k - n$; $h(P) = 1$ if P is distinct from but collinear with C; $h(P) = 0$ otherwise. If a is a line through C, h sums, over a, to $k - n + n - 1 = k - 1$. If a is a line not through C, the $k-1$ lines through C which are not in the same class as

a meet a in $k-1$ distinct points; hence h sums to $k-1$ over a in this case also. Therefore h represents $k-1$ on N.

(iv) By (ii) and (iii), $\phi(N)$ divides n, $k-1$ and their greatest common divisor $(n, k-1)$.

(v) Let $\phi(N)$ be represented by f on the affine plane N, and let s be the sum of f over the n^2 points of N. Considering the sum of the sums of f over the n lines of some class, we find $n\phi(N) = s$. On the other hand, if C is a point of N, every point of N (other than C) lies on exactly one of the $n+1$ lines through C. Considering the sum of the sums over the $n+1$ lines through C, we find $nf(C) + s = (n+1)\phi(N)$. Since $s = n\phi(N)$, $nf(C) = \phi(N)$. Therefore $n|\phi(N)|n$, so $\phi(N) = n$. And, incidentally, $f(C) = 1$ for every point C of N.

(vi) In view of (ii), every positive integral multiple of n is positively represented on N. Next let r be an integer divisible by $\phi(N)$, in the range $0 < r < n$. Certainly r is represented on N by some function f. Let m' be the least value assumed by f. Then, if f' is the function defined in (ii) and if m is any integer satisfying $m \geqslant -m'$, the integer $r+mn$ is positively represented on N by $f+mf'$. Therefore, in every congruence class of integers mod n divisible by $\phi(N)$, there is at most a finite number of positive integers not represented positively on N.

This completes the proof of Theorem 1. The Corollary follows from (i).

3. A characterization of ϕ. If N is a net of degree k, order n, we shall assume henceforth that the k classes of "parallel" lines have been numbered (arbitrarily, but once and for all) from 1 to k. Thus, if $1 \leqslant i \leqslant k$, an i-line of N is a line of class i. In terms of an arbitrary "centre" C (C a point of N) we introduce a coordinate system as follows: For $1 \leqslant i \leqslant k$, the n lines of class i are numbered from 1 to n, the i-line through C being assigned the number 1. The i-line numbered x is designated by (i, x). We also introduce k point-functions I_i, the indicators, by defining $I_i(P) = x$ if (i, x) is the i-line through the point P.

If f is a single-valued function from the integer-range $1 \leqslant x \leqslant n$ to the integers, we shall designate by $f(*)$ the sum $f(1) + f(2) + \ldots + f(n)$. In terms of these notations we may prove two theorems.

THEOREM 2. *Let N be a net of degree k, order n. Then a necessary and sufficient condition that the integer m be represented on N is that m be represented* mod n *on N.*

THEOREM 3. *Let N be a net of degree k, order n. Then $\phi(N)$ is the smallest positive integer s with the following property: If $f_1, \ldots, f_k$ are single-valued functions from the integer-range $1 \leqslant x \leqslant n$ to the integers, such that*

$$f_i(1) \equiv 0 \bmod n \qquad (i = 1, \ldots, n), \tag{1}$$

$$\sum_{i=1}^{k} f_i(I_i(P)) \equiv 0 \bmod n \tag{2}$$

for each point P of N, then

$$sf_1(*) \equiv 0 \bmod n.$$

Proof. If $a_1, \ldots, a_{kn}$ are the kn lines and $P_1, \ldots, P_{n^2}$ are the n^2 points of N, in arbitrary arrangements, define the *line-point incidence matrix* A of N by putting 1 or 0 in the uth row, vth column of A according as P_v does or does not lie on a_u. Also define U to be the column vector of order kn with every element 1. Let X be a column vector of order n^2 and let m be an arbitrary integer. Then m is represented on N if and only if

$$AX = mU \tag{3}$$

for an integral X. In view of Theorem 1 (ii), (3) has a *rational* solution X with every component equal to m/n. If $r = \text{rank } A$, there exist unimodular matrices T, Q (with rational integral components) such that

$$TAQ = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_r = \text{diag}(e_1, e_2, \ldots, e_r), \tag{4}$$

where the positive integers e_j are the *invariant divisors* of A; thus $e_j | e_{j+1}$ for $j = 1, 2, \ldots, r-1$. Setting $TU = V$, $X = QY$, we see that (3) may be reduced to

$$e_j y_j = m v_j \qquad (j = 1, \ldots, r). \tag{5}$$

A necessary and sufficient condition that (3) have an integral solution X is that (5) yield integral values for $y_1, \ldots, y_r$. In particular, by the definition of $\phi(N)$, if

$$d_j = (e_j, v_j), \quad e_j = h_j d_j \qquad (j = 1, \ldots, r), \tag{6}$$

then $\phi(N)$ is the least common multiple

$$\phi(N) = [h_1, \ldots, h_r]. \tag{7}$$

Next let u be any integer divisible by e_r (and hence by each e_j.) Clearly m is represented mod u on N if and only if $AX \equiv mU \bmod u$ for an integral X, or, equivalently, if and only if $e_j y_j \equiv m v_j \bmod u$ for integral y_j ($j = 1, \ldots, r$). Since $e_j | u$, the latter congruences imply $e_j | m v_j$, $h_j | m$, $\phi(N) | m$. However, if $\phi(N) | m$, m is certainly represented on N. Thus Theorem 2 will be proved when we show that $e_r | n$.

For $i = 1, \ldots, k$, let the row-vector R_i denote the sum of the n rows of A corresponding to the lines of class i. Since each point lies on exactly one i-line R_i has each component equal to 1; thus $R_1 = R_2 = \ldots = R_k$. Let B be the matrix of $1 + k(n-1)$ rows obtained by deleting from A the rows corresponding to the 2-line, 3-line, ..., k-line through the centre C. Clearly, since $R_i = R_1$, $T'A = \begin{pmatrix} B \\ 0 \end{pmatrix}$ for a unimodular matrix T'; hence B has the same rank and invariant divisors as A. There is therefore no loss of generality in assuming that, in (4), the first r rows of T have zeros in the columns matching with the $k-1$ rows of A omitted in B. With this understanding, let V_j be the jth row of T ($j = 1, \ldots, r$); by (4), since Q is unimodular, e_j is the greatest

common divisor of the components of V_jA. For any fixed j, let $g_i(x)$ denote the component of V_j in the column corresponding to the line (i, x) of N; thus $g_i(1) = 0$ for $i > 1$. In V_jA, the column corresponding to point the P has component

$$\sum_{i=1}^{k} g_i(I_i(P)) \equiv 0 \bmod e_j. \tag{8}$$

When $P = C$, (8) reduces to $g_1(1) \equiv 0 \bmod e_j$; hence

$$g_i(1) \equiv 0 \bmod e_j \qquad (i = 1, \ldots, k). \tag{9}$$

Selecting a fixed line (i, x) and summing the congruence (8) over the n points P of (i, x), we derive

$$\sum_{u \neq i} g_u(*) + ng_i(x) \equiv 0 \bmod e_j. \tag{10}$$

From (10), (9), $ng_i(x) \equiv ng_i(1) \equiv 0 \bmod e_j$. Thus, if $d = (n, e_j)$ and $e_j = de'$, we have $g_i(x) \equiv 0 \bmod e'$ for all i, x. Since T is unimodular, the greatest common divisor of the components of V_j is 1; therefore $e' = 1$ and $e_j | n$. In particular $e_r | n$, proving Theorem 2.

In similar fashion, letting the g_i be arbitrary rational-valued functions such that $g_i(1) = 0$ for $i > 1$, and replacing the congruences (8) by equations, we may deduce that $g_i(x) = 0$ for all i, x. This shows that the rows of B are linearly independent, so that

$$r = 1 + k(n - 1). \tag{11}$$

To prove Theorem 3, let V_j have (integer-valued) components $g_i(x)$, as above, and let $f_i(x) = n_j g_i(x)$ where $n = n_j e_j$. Then (9) and (8) become (1) and (2) respectively. On the other hand, $V_jU = v_j$, in the notation of (5), and hence

$$\sum_{i=1}^{k} f_i(*) = n_j v_j. \tag{12}$$

Multiplying (5) by n_j, we get $ny_j = m.n_jv_j$. Therefore m is represented on N if and only if $m.n_jv_j \equiv 0 \bmod n$ for $j = 1, \ldots, r$. To replace (10) we have $\sum_{u \neq i} f_u(*) \equiv 0 \bmod n$, whence, by (12), $n_jv_j \equiv f_i(*)$ for $i = 1, \ldots, k$. In particular, $n_jv_j \equiv f_1(*) \bmod n$. Thus, by the definition of s, $s.n_jv_j \equiv sf_1(*) \equiv 0 \bmod n$, for $j = 1, \ldots, r$. Hence s is represented on N, $\phi(N) | s$.

We must prove the converse. Certainly $AX = \phi(N)U$ for an integral X. Let $f_1, \ldots, f_k$ be integer-valued functions satisfying (1) and (2), and let V be the row-vector with $f_i(x)$ in the column corresponding to line (i, x). Then VA has component $\sum_{i=1}^{k} f_i(I_i(P))$ in the column corresponding to point P, while $VU = \sum_{i=1}^{k} f_i(*)$. Thus the equation $VAX = \phi(N)VU$, together with the congruences (2), implies that

$$\phi(N) \sum_{i=1}^{k} f_i(*) \equiv 0 \bmod n. \tag{13}$$

By the same methods as before, we deduce from (1) and (2) that (13) is equivalent to $\phi(N)f_1(*) \equiv 0 \bmod n$. Since s is the least positive integer such that $sf_1(*) \equiv 0 \bmod n$ for all such functions f_i, $s \mid \phi(N)$. Therefore $\phi(N) = s$. This completes the proofs of Theorems 2, 3.

3. Direct products of nets. Let N_1, N_2 be nets of orders n_1, n_2 respectively, and of the same degree k. The *direct product* $N = N_1 \times N_2$ is defined as follows: (i) The points of N are the ordered pairs (P_1, P_2), with P_j a point of N_j. (ii) For $i = 1, \ldots, k$, the i-lines of N are the ordered pairs (a_1, a_2), with a_j an i-line of N_j. (iii) (P_1, P_2) lies on (a_1, a_2) in N if and only if P_j lies on a_j in N_j for $j = 1, 2$. It is easy to verify that N is a net of degree k, order n_1n_2. Making the obvious identifications one may establish the commutative and associative laws for direct products.

If N_1 has a coordinate system centered about C_1, with indicators I_i, and N_2 has a coordinate system centered about C_2, with indicators J_i, we introduce a natural coordinate system for $N = N_1 \times N_2$ as follows: Take $C = (C_1, C_2)$ as centre. If a_j is the i-line (i, x_j) of N_j $(j = 1, 2,)$ denote by $(i; x_1, x_2)$ the i-line (a_1, a_2) of N. Define the indicators I_i of N by $I_i(P_1, P_2) = (x_1, x_2)$ where $(i; x_1, x_2)$ is the i-line of N through (P_1, P_2). Moreover, if $f(x_1, x_2)$ is a function from the integer-domain $1 \leqslant x_1 \leqslant n_1$, $1 \leqslant x_2 \leqslant n_2$ to the integers, denote by $f(*, x_2)$ the sum $f(1, x_2) + f(2, x_2) + \ldots + f(n_1, x_2)$. Similar meanings are assigned to $f(x_1, *)$ and $f(*, *)$.

THEOREM 4. *Let N_j be a net of order n_j and degree k, for $j = 1, 2$, and let $N = N_1 \times N_2$. Write*

$$d = (n_1, n_2), \qquad n_1 = dq_1, \qquad n_2 = dq_2. \tag{14}$$

Then there exist positive integers a, b such that

$$(q_1, \phi(N_1)) \cdot (q_2, \phi(N_2)) = a \cdot \phi(N), \tag{15}$$

$$(d, k - 1) \cdot \phi(N) = b\,[\phi(N_1), \phi(N_2)], \tag{16}$$

$$ab \mid (d, k - 1). \tag{17}$$

COROLLARY 1. *If $(n_1, n_2, k - 1) = 1$, then $\phi(N_1 \times N_2) = \phi(N_1)\phi(N_2)$.*

COROLLARY 2. *If $(q_1q_2, k - 1) = 1$, then $\phi(N_1 \times N_2) = 1$.*

COROLLARY 3. *For any finite net N, $\phi(N \times N) = 1$.*

Proof. In the present notation the content of Theorem 3 may be expressed as follows: $\phi(N)$ is the least positive integer such that, for integer-valued functions f_i, the congruences

$$f_i(1, 1) \equiv 0 \bmod n_1n_2, \tag{18}$$

$$\sum_{i=1}^{k} f_i(I_i(P_1), J_i(P_2)) \equiv 0 \bmod n_1n_2 \tag{19}$$

for all points (P_1, P_2) of N, imply

$$\phi(N) f_1(*, *) \equiv 0 \bmod n_1 n_2. \tag{20}$$

Keeping P_1 fixed in (19), select a line (i, x_2) of N_2 and sum over all points P_2 of (i, x_2). Then

$$\sum_{j \neq i} f_j(I_j(P_1), *) + n_2 f_i(I_i(P_1), x_2) \equiv 0 \bmod n_1 n_2. \tag{21}$$

Since the sum in (21) is independent of x_2, we have

$$n_2 f_i(I_i(P_1), x_2) \equiv n_2 f_i(I_i(P_1), 1) \bmod n_1 n_2,$$

i.e.

$$f_i(x_1, x_2) \equiv f_i(x_1, 1) \bmod n_1 \tag{22}$$

for all i, x_1, x_2 in their respective ranges. Similarly,

$$f_i(x_1, x_2) \equiv f(1, x_2) \bmod n_2. \tag{23}$$

Since d divides n_1, n_2, we deduce from (22), (23) and (18) that

$$f_i(x_1, x_2) \equiv 0 \mod d. \tag{24}$$

Returning to (21), choose any line (j, x_1) of N_1, with $j \neq i$, and sum over all points P_1 of (j, x_1). There results

$$\sum_{p \neq i, j} f_p(*, *) + n_1 f_j(x_1, *) + n_2 f_i(*, x_2) \equiv 0 \bmod n_1 n_2 \tag{25}$$

for all i, j $(i \neq j)$ and x_1, x_2. As in the proof of Theorem 3, $f_p(*, *) \equiv f_1(*, *) \bmod n_1 n_2$. And since, by Theorem 1, $\phi(N)$ divides $k - 1$, $(k - 1) f_1(*, *) \equiv 0 \bmod n_1 n_2$. Therefore (25) is equivalent to

$$f_1(*, *) \equiv n_1 f_j(x_1, *) + n_2 f_i(*, x_2) \bmod n_1 n_2. \tag{26}$$

Since $k \geqslant 3$, and since in (26) the only restriction is $i \neq j$, (26) is equivalent to

$$f_1(*, *) \equiv n_1 f_1(x_1, *) + n_2 f_1(*, x_2) \bmod n_1 n_2. \tag{27}$$

Define t_1, t_2 as the least positive integers such that

$$t_1 n_2 f_1(*, x_2) \equiv 0, \qquad t_2 n_1 f_1(x_1, *) \equiv 0 \qquad \bmod n_1 n_2 \tag{28}$$

for all f_i satisfying (18), (19). By (27), $t_1 t_2 f_1(*, *) \equiv 0 \bmod n_1 n_2$. Hence, by the property (20) of $\phi(N)$,

$$\phi(N) \mid t_1 t_2. \tag{29}$$

Since $q_1 d = n_1$, (24) implies $q_1 n_2 f_1(*, x_2) \equiv 0 \bmod n_1 n_2$. Thus $t_1 \mid q_1$. Similarly,

$$t_j \mid q_j \qquad (j = 1, 2). \tag{30}$$

Since the q_j are relatively prime, so are the t_j. Next choose any fixed value

for x_2 and define functions $F_i(x_1) = f_i(x_1, x_2)$. From (22), $F_i(x_1) \equiv f_i(x_1, 1) \bmod n_1$. Thus, from (18), $F_i(1) \equiv 0 \bmod n_1$. Moreover, by (19),

$$\sum_{i=1}^{k} F_i(I_i(P_1)) \equiv \sum_{i=1}^{k} f_i(I_i(P_1), J_i(C_2)) \equiv 0 \bmod n_1.$$

Therefore, by Theorem 3, $0 \equiv \phi(N_1)F_1(*) \equiv \phi(N_1)f_1(*, x_2) \bmod n_1$, and so $\phi(N_1)n_2f_1(*, x_2) \equiv 0 \bmod n_1n_2$. Hence (and similarly)

$$t_j \mid \phi(N_j) \qquad (j = 1, 2). \tag{31}$$

By (30), (31), t_j divides the greatest common divisor of q_j and $\phi(N_j)$. Hence (29) implies (15) for some positive integer a.

To obtain (16), let $g_i(x_1)$ be any set of integer-valued functions satisfying equations analogous to (1), (2) for N_1, and set $f_i(x_1, x_2) = n_2g_i(x_1)$. Then the f_i will satisfy (18), (19). Therefore $\phi(N)f_1(*, *) = \phi(N)(n_2)^2g_1(*) \equiv 0 \bmod n_1n_2$, $\phi(N)n_2g_1(*) \equiv 0 \bmod n_1$, $\phi(N_1) \mid \phi(N)n_2$. Since $\phi(N_1) \mid (n_1, k-1)$ and since $(n_1, n_2) = d$, we may improve the last statement to $\phi(N_1) \mid (d, k-1)\phi(N)$. Similarly for $\phi(N_2)$. Hence the least common multiple $[\phi(N_1), \phi(N_2)]$ divides $(d, k-1)\phi(N)$, proving (16) for some positive integer b.

Eliminating $\phi(N)$ from (15), (16), we derive $ab[\phi(N_1), \phi(N_2)] = (d, k-1)(q_1, \phi(N_1))(q_2, \phi(N_2))$. Since the integers $(q_j, \phi(N_j))$ are relatively prime divisors of $[\phi(N_1), \phi(N_2)]$, we have (17). This completes the proof of Theorem 4. In the case of Corollary 1, $a = b = 1$, by (17), and then $\phi(N) = \phi(N_1)\phi(N_2)$ by (16) and the fact that $(\phi(N_1), \phi(N_2))$ is a divisor of $(n_1, n_2, k-1) = 1$.

In the case of Corollary 2, the left-hand side of (15) is 1, since, for example, $(q_1, \phi(N_1))$ divides $(q_1, k-1) = 1$. Thus $\phi(N) = 1$. And Corollary 3 corresponds to the special case $q_1 = 1 = q_2$ of Corollary 2.

THEOREM 5. *Let $n > 1$ be a positive integer with factorization $n = \Pi p(i)^{m(i)}$ where the $p(i)$ are distinct primes and the $m(i)$ are positive integers. Let $r = \min(p(1)^{m(1)}, p(2)^{m(2)}, \ldots)$. Then there exists a net N of order n, degree $r+1$, such that $\phi(N) = r > 1$. In particular, no line can be adjoined to N.*

COROLLARY. *If k is any integer such that $3 \leqslant k \leqslant r+1$, there exists a net N of order n, degree k.*

Proof. For any prime p and positive integer m, let $\mathrm{E}(p, m)$ be an affine plane of order p^m (and degree $p^m + 1$.) Such a plane exists, for example, the plane obtained by using coordinates (in the familiar manner of elementary plane geometry) from the field $\mathrm{GF}(p^m)$. For each i, we may define a net N_i of degree $r+1$, order $p(i)^{m(i)}$ from an $\mathrm{E}(p(i), m(i))$ by deleting some $(p(i)^{m(i)}+1)-(r+1)$ classes of lines. Set $N = N_1 \times N_2 \times \ldots$. By an obvious extension of Corollary 1 to Theorem 4, $\phi(N) = \phi(N_1)\phi(N_2)\ldots$. For exactly one i, $N_i = \mathrm{E}(p(i), m(i))$ and $\phi(N_i) = p(i)^{m(i)} = r$. For all other i, lines can be adjoined to N_i, so $\phi(N_i) = 1$. Therefore $\phi(N) = r > 1$. As for the Corollary we need merely delete some $r + 1 - k$ classes of lines from N.

4. Homomorphic nets. Let N, N' be nets of the same degree k. A *homomorphism* θ of N upon N' is a single-valued, exhaustive mapping of N upon N' which maps points upon points, i-lines upon i-lines (for $i = 1, \ldots, k$) and preserves incidence. The requirement that i-lines be mapped upon i-lines may seem artificial. The obvious generalization, however, is no more necessary than the little used concept of "anti-homomorphism" in group theory, and adds complications to the proofs. (See Bates [1] for a similar restriction in regard to 3-nets.)

A homomorphism θ of N upon N' is called an *isomorphism* if it is one-to-one, and a *zero homomorphism* if N' has order one. A net N is *simple* if its only homomorphisms upon nets are isomorphisms and zero homomorphisms.

LEMMA 1. *Let N, N' be nets of respective orders n, n' and of the same degree k. Let θ be a homomorphism of N upon N'. For each point P' of N', let $M(P')$ be the subset of N consisting of all points P of N such that $P\theta = P'$ and of all lines a of N such that $a\theta$ passes through P'. Then $n = mn'$ for a positive integer m, and each $M(P')$ is a subnet of N, of order m, degree k.*

COROLLARY. *Every finite affine plane is a simple net.*

Proof. Consider one of the sets $M = M(P')$. Then M contains lines of each of the k classes in N, since the k lines through P' are images under θ. If a, b are lines of distinct classes in N, such that $a\theta$, $b\theta$ pass through P', the intersection point $P = a \cdot b$ satisfies $P\theta = P'$, and hence is in M. If Q is in M, each of the k lines through Q is in M. Hence M is a net of degree k and of some order m. In particular, for each i, M has exactly m i-lines, and these are precisely the i-lines of N which map into the i-lines through P'. If Q' is a point of N', distinct from P', the i-line through P' and the j-line through Q' ($j \neq i$) must meet in a point R' of N'. Then $M(R')$, $M(P')$ have the same i-lines, hence the same order m; and $M(Q')$, $M(R')$ have the same j-lines, hence the same order m. Therefore each of the $(n')^2$ subnets $M(P')$ has order m, showing that $(n')^2m^2 = n^2$ or $n = mn'$.

As for the Corollary, if the net N has order $n = k - 1$, then $k - 1 = mn'$. But either $n' = 1$ or $n' \geqslant k - 1$; and the second alternative gives $n' = k - 1$, $m = 1$. Hence every homomorphism of N upon a net is either a zero homomorphism or an isomorphism. Thus N is simple. This Corollary offers a partial explanation of the lack of success in attempting to define homomorphisms of projective planes (Marshall Hall [1]).

With the notation of Lemma 1, define D to be the greatest common divisor of all the integers $\phi(P') = \phi(M(P'))$. Also write

$$d = (m, n'), \qquad m = du, \qquad n' = dv. \tag{32}$$

THEOREM 6. *Let N be a net of degree k, order $n = mn'$, possessing a proper homomorphism θ upon a net N' of order n'. (Thus $m, n' \geqslant k - 1$.) Then*

$$\phi(N) \mid [\phi(N'), (u, D)], \qquad \phi(N') \mid (d, k - 1)\phi(N). \tag{33}$$

Proof. If $\phi(N)$ is represented on N by the point function $f(P)$, define $g(P') = \sum f(P)$ where the sum is taken over the m^2 points P such that $P\theta = P'$. Then it is easy to see that g represents $m\phi(N)$ on N'. Hence $\phi(N') \mid m\phi(N)$. By (32) and the fact that $\phi(N') \mid (n', k-1)$, we deduce the second relation of (33). If a, b, c are integers, one readily verifies the identity $([a, b], [a, c]) = [a, (b, c)]$. Thus the relation

$$\phi(N) \mid [\phi(N'), (u, \phi(C'))], \tag{34}$$

holding for every point C' of N', implies the first relation of (33). We complete the proof by establishing (34).

Let θ' be any one-to-one mapping of the points of N' into the points of N, such that $P'\theta'\theta = P'$. Thus $P'\theta'$ is in $M(P')$ for each P' of N'. Choose any point C' as centre in N' and take $C = C'\theta'$ as centre in N. Let I_i, J_i be the indicator functions for N, N' respectively. As an additional notation, define

$$I_i(P') = I_i(P'\theta'), \qquad P' \text{ in } N'. \tag{35}$$

If $f(x)$ is a function from the integer-range $1 \leqslant x \leqslant n$ to the integers, define $f(*)$ as before. Also define

$$f(i, P') = \sum{}' f(x), \tag{36}$$

where the sum in (36) is taken over all x such that (i, x) is a line of $M(P')$. Now let f_i be functions satisfying (1), (2) of Theorem 3. For any point P' of N', and any line (i, x) of $M(P')$, sum (2) over all points P common to (i, x) and $M(P')$. Thus

$$\sum_{j \neq i} f_j(j, P') + mf_i(x) \equiv 0 \bmod n. \tag{37}$$

Since the second term of (37) is independent of the choice of (i, x) in $M(P')$,

$$f_i(x) \equiv f_i(I_i(P')) \bmod n', \qquad (i, x) \text{ in } M(P'). \tag{38}$$

By (38), $f_i(x)$ is determined mod n' by the line $(i, x') = (i, x)\theta$ of N'. Thus (mod n') we may define a set of integer-valued functions $F_i(x')$, on the range $1 \leqslant x' \leqslant n'$, by

$$F_i(x') \equiv f_i(x) \bmod n' \qquad \text{if } (i, x)\theta = (i, x'). \tag{39}$$

Clearly the F_i satisfy the conditions corresponding to (1), (2) for N'. Therefore, by Theorem 3,

$$\phi(N')F_1(*) \equiv 0 \bmod n'. \tag{40}$$

Next pick $j \neq i$ and consider the line $(j, 1)$ of N'. By (39), (38),

$$F_i(*) \equiv \sum{}' f_i(I_i(P')) \bmod n' \tag{41}$$

where the sum in (41) is over the points P' of $(j, 1)$. Moreover $f_j(j, P') = f_j(j, C')$ for P' on $(j, 1)$, since, for each P' of $(j, 1)$, the j-lines of $M(P')$ are

those lines (j, x) such that $(j, x)\theta = (j, 1)$. Hence, if in (37) we sum over all points P' of $(j, 1)$, there results

$$\sum_{p \not\equiv i, j} f_p(*) + n' f_j(j, C') + m F_i(*) \equiv 0 \mod n. \tag{42}$$

As in the proof of Theorem 5, (42) is equivalent to

$$f_1(*) \equiv n' f_1(1, C') + m F_1(*) \mod n. \tag{43}$$

If t is the least positive integer such that

$$t n' f_1(1, C') \equiv 0 \mod n \tag{44}$$

for all functions f_i satisfying (1), (2), then (40), (43) imply $[\phi(N'), t] f_1(*) \equiv 0 \mod n$. Hence

$$\phi(N) \mid [\phi(N'), t]. \tag{45}$$

Since C is the centre for $M(C')$ as well as for N, and since $m \mid n$, the f_i satisfy conditions analogous to (1), (2) for the net $M(C')$. And since $f_1(1, C')$ denotes for $M(C')$ the sum analogous to $f_1(*)$ for N, $\phi(C') f_1(1, C') \equiv 0 \mod m$. Inasmuch as $n = mn'$, this and (44) imply $t \mid \phi(C')$. Again, from (38), if $(1, x)$ is in $M(C')$, $f_1(x) \equiv f_1(I_1(C')) \equiv f_1(1) \equiv 0 \mod n'$ and therefore $f_1(1, C') \equiv 0 \mod n'$. Moreover, $un' \cdot n' = nv$, by (32), so that $un' f(1, C') \equiv 0 \mod n$. Hence $t \mid u$. Therefore

$$t \mid (u, \phi(C')). \tag{46}$$

And (45), (46) combine to give (34). This completes the proof of Theorem 6.

5. Explicit evaluation of ϕ for nets of degree 3. A set G together with an operation $(.)$ is called a *iwop* provided: (i) if a, b are in G, $a \cdot b$ is a uniquely determined element of G; (ii) if a, b are in G there exists a unique x in G such that $x \cdot a = b$ and a unique y in G such that $a \cdot y = b$; (iii) there exists a (unique) element 1 of G such that $a \cdot 1 = 1 \cdot a = a$ for every a in G. A loop is a group if and only if it obeys the associative law $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. The concepts of homomorphism, normal subloop and quotient loop are quite similar to the corresponding concepts in group theory (Albert [1], Baer [2], Bruck [1]). For our purposes the essential facts are these: If the loop G of order n possesses a homomorphism θ upon a loop G' of order n', then the kernel H of θ is a normal subloop of G and G/H is isomorphic to G'. Moreover H has order m where $n = mn'$, and each element of G' is the image under θ of precisely m distinct elements of G. Finally, there is a one-to-one correspondence between the normal subloops of G and the homomorphisms of G upon loops.

From a loop G of order n, we form a net $N = N(G)$ of order n, degree 3, as follows: The points of N are the n^2 ordered pairs (x, y) of elements of G. Each a of G determines: (i) a 1-line $x = a$ whose points are the n points (a, y); (ii) a 2-line $y = a$ whose points are the n points (x, a); (iii) a 3-line $x \cdot y = a$

whose points are the n points (x, y) with $x \cdot y = a$. As shown in Bates [1], every net of order n, degree 3 may be so defined in terms of a suitable loop G of order n. We define $\phi(G) = \phi(N(G))$.

THEOREM 7. *Let G be a finite loop of order n. If G contains a normal subloop H of odd order such that the quotient loop G/H is a cyclic group of even order, then $\phi(G) = 2$. In all other cases, $\phi(G) = 1$.*

COROLLARY 1. *Necessary and sufficient conditions that $\phi(G) = 2$ are that $n = m \cdot 2^t$ for m odd, $t \geqslant 1$, that G contain a normal subloop K of order m, and that G/K be the cyclic group of order 2^t.*

COROLLARY 2. *If $n = 4m + 2$, then $\phi(G) = 2$ if and only if G contains a subloop of order $2m + 1$.*

COROLLARY 3. *If G is a group of order $n = 4m + 2$, then $\phi(G) = 2$.*

Proof. Take $C = (1, 1)$ as the centre of the net $N(G)$ and define indicators $I_i (i = 1, 2, 3)$ so that if $P = (x, y)$ then $I_1(P) = x$, $I_2(P) = y$, $I_3(P) = x \cdot y$. Conditions (1), (2) of Theorem 3 become

$$f_1(1) \equiv f_2(1) \equiv f_3(1) \equiv 0 \bmod n, \tag{47}$$

$$f_1(x) + f_2(y) + f_3(x \cdot y) \equiv 0 \bmod n, \tag{48}$$

for all x, y of G. Setting, in turn, $x = 1$ and $y = 1$ in (48), we find, by (47), that $-f_3(x) \equiv f_1(x) \equiv f_2(x) \equiv f(x)$, say, mod n so that (47), (48) can be replaced by

$$f(x \cdot y) \equiv f(x) + f(y) \bmod n \tag{49}$$

for all x, y of G. In view of (49), the mapping $x \to f(x)$ is a homoorphism of G upon some subgroup Z of the additive group of the integers mod n. Thus Z is a cyclic group.

Conversely, if G is homomorphic to a cyclic group Z of order n', we may assume without loss of generality that Z is a subgroup of the additive group of integers mod n and that the homomorphism is given by (49). Also $n = mn'$ where m is the order of the kernel, and exactly m elements of G map upon each element of Z. If t is the sum of the elements of Z it is easily verified (compare Paige [1]) that t is the unit 0 if n' is odd and the unique element of order two if n' is even. In any case, by Theorem 3, $\phi(G)$ is the least positive integer s such that

$$sf(*) \equiv smt \equiv 0 \bmod n \tag{50}$$

for all integer-valued f satisfying (49). Clearly $\phi(G) \mid 2$.

If n' is even, t has order two. If also m is odd, (50) implies that $2 \mid \phi(G)$. Therefore $\phi(G) = 2$, proving the first statement of Theorem 7.

Next suppose that G is such that there exists no f for which m is odd and n'

is even. If m is odd, n' is odd and $t \equiv 0$, so that $f(*) \equiv 0 \bmod n$. If m is even, $f(*) \equiv mt \equiv 0$, since $2t \equiv 0 \bmod n$. Therefore $\phi(G) = 1$. This completes the proof of Theorem 7. The Corollaries are immediate consequences of known facts about loops and groups.

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Vol. II, No. 2: Errata

WATER WAVES OVER A CHANNEL OF FINITE DEPTH

ALBERT E. HEINS

p. 216: Instead of $L_+(w)$, read $1/L_+(w)$. Also, for $\exp[\chi(w)]$ in this expression read $\exp[-\chi(w)]$.

p. 221: Multiply $|L_+(\pm\kappa)|^2$ by b.

Multiply $|L_+(\pm\kappa')|^2$ by $b\rho_0$.

Divide the expression for $\left|\dfrac{L_+(\pm\kappa)}{L_+(\pm\kappa')}\right|^2$ by ρ_0^2.

In the formulas for t_1 and t_2, $\kappa\rho_0/2$ and $2\kappa'/\rho_0$ should be replaced by 2κ and $2\kappa'$, respectively.

GREEN'S FORMS AND MEROMORPHIC FUNCTIONS ON COMPACT ANALYTIC VARIETIES

KUNIHIKO KODAIRA

1. Introduction. Let $\mathfrak{M}$ be a compact complex analytic variety of the complex dimension n with a positive definite Kählerian metric [4] ; the local analytic coordinates on $\mathfrak{M}$ will be denoted by $z = (z^1, z^2, \ldots, z^n)$. Now, suppose a meromorphic function $f(z)$ defined on $\mathfrak{M}$ as given. Then the poles and zero-points of $f(z)$ constitute an analytic surface[1] in $\mathfrak{M}$ consisting of a finite number of irreducible closed analytic surfaces $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$, each of which is a polar or a zero-point variety of $f(z)$. The formal sum $D = \Sigma m_k \Gamma_k$ of these varieties multiplied respectively by the multiplicity m_k of Γ_k is called the *divisor* of $f(z)$, where the multiplicities of the polar varieties are to be associated with the negative sign. The divisor D of $f(z)$ can be also defined in case $f(z)$ is a many-valued meromorphic function, if the absolute value $|f(z)|$ is one-valued. Such a function $f(z)$ will be called *multiplicative*, since, if one prolongs $f(z)$ analytically along a closed continuous curve ζ, then $f(z)$ is multiplied by a constant factor $\chi(\zeta)$ of modulus 1 depending only on the homology class of ζ on $\mathfrak{M}$. From the topological viewpoint, the divisor D is a $(2n-2)$-cycle on $\mathfrak{M}$. It can be readily verified that the divisor D of an arbitrary multiplicative meromorphic function must be a bounding cycle.[2] Obviously a multiplicative meromorphic function is determined by its divisor uniquely up to a non-vanishing multiplicative constant. Now, assume conversely that a bounding $(2n-2)$-cycle $D = \Sigma m_k \Gamma_k$ consisting of a finite number of irreducible closed analytic surfaces $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ is given. Then, *does a multiplicative meromorphic function having D as its divisor exist?* This question of fundamental importance was solved affirmatively by A. Weil in a more general form.[3] His skilful method of proof of the existence is based on the theory of harmonic integrals but entirely differs from the classical potential theoretical treatment of the problem in the case of Riemann surfaces.

In the present paper we shall prove the existence of the multiplicative meromorphic function with the given divisor by a *potential-theoretical method* and give simultaneously *an explicit expression of the multiplicative meromorphic function in terms of the integral of the Green's form extending over the given divisor* [10, pp. 140-143]. In order to explain our main idea more explicitly, let us first

Received December 9, 1949.

[1] By an analytic surface we shall mean a $(n-1)$-dimensional analytic subvariety of $\mathfrak{M}$.

[2] By a bounding cycle we shall mean a cycle which is the boundary of a chain with real coefficients.

[3] Weil [12]. As to the special case $n = 2$, this question was solved also by the author in Japan independently of the results of A. Weil; see Kodaira [6].

consider a compact Riemann surface $\mathfrak{R}$ with the metric $ds^2 = g^{\frac{1}{2}}|dz|^2$, where $z = x + iy$ denote local uniformization variables on $\mathfrak{R}$. Choose two different points $\mathfrak{p}$, $\mathfrak{q}$ on $\mathfrak{R}$ arbitrarily and suppose that $\mathfrak{p}$ or $\mathfrak{q}$ carries respectively unit "charge" of positive or negative sign. The "electric potential" $\gamma(z, \mathfrak{p}, \mathfrak{q})$ produced by these two charges is a harmonic function of z having logarithmic singularities at $\mathfrak{p}$ and $\mathfrak{q}$, whose existence is proved by the Dirichlet principle [13]. Now, let a divisor $\mathfrak{d} = \Sigma m_k \mathfrak{p}_k$ of total order 0 be given, and consider the sum

$$\gamma[\mathfrak{d}](z) = \Sigma m_k \gamma(z, \mathfrak{p}_k, \mathfrak{q})$$

of the potential $\gamma(z, \mathfrak{p}, \mathfrak{q})$ extending over the divisor $\mathfrak{d}$. Then, denoting by $*d\gamma$ the dual form $\gamma_y dx - \gamma_x dy$ of $d\gamma = \gamma_x dx + \gamma_y dy$, the multiplicative meromorphic function with the divisor $\mathfrak{d}$ is given by the formula

$$f(z) = c. \exp\left\{ -\gamma[\mathfrak{d}](z) + i \int^z *d\gamma[\mathfrak{d}](z) \right\}, \tag{1.1}$$

where c means a non-vanishing constant [13, §17]. In the above construction of $f(z)$, the negative charge at $\mathfrak{q}$ plays merely the role of *compensating term* in order to make the total sum of the charges over $\mathfrak{R}$ equal to zero and is cancelled in making the sum $\gamma[\mathfrak{d}](z)$. Hence, replacing it by the charge of the total magnitude -1 distributed uniformly over the whole space $\mathfrak{R}$, we can eliminate the auxiliary point $\mathfrak{q}$ from our whole construction. This leads to replacing $\gamma(z, \mathfrak{p}, \mathfrak{q})$ by its *mean*

$$\gamma(z, \mathfrak{p}) = \sigma^{-1} \iint \gamma(z, \mathfrak{p}, \mathfrak{q}) d\sigma_{\mathfrak{q}},$$

where $d\sigma_{\mathfrak{q}}$ denotes the surface element $g^{\frac{1}{2}}dxdy$ and σ is the total area of $\mathfrak{R}$. $\gamma(z, \mathfrak{p})$ thus obtained will be called *the Green's function for the compact Riemann surface* $\mathfrak{R}$. *The Green's function is not harmonic but satisfies the inhomogeneous Laplace equation*

$$\Delta\gamma(z, \mathfrak{p}) = 2\pi g^{\frac{1}{2}}\sigma^{-1}$$

and has the typical singularity $-\log|z - z_{\mathfrak{p}}|$ *at* $\mathfrak{p}$. *By using the Green's function, the sum* $\gamma[\mathfrak{d}](z)$ *is represented as*

$$\gamma[\mathfrak{d}](z) = \Sigma m_k \gamma(z, \mathfrak{p}_k), \tag{1.2}$$

and thus the auxiliary point $\mathfrak{q}$ has been eliminated.

Now we turn to the compact analytic variety $\mathfrak{M}$ of an arbitrary dimension n with a positive definite Kählerian metric. Considering $\mathfrak{M}$ as a Riemannian variety, we have, in $\mathfrak{M}$, the *Green's form* $\gamma^\rho(z, \mathfrak{p})$ of any rank $\rho, 0 \leqslant \rho \leqslant 2n$, introduced by G. de Rahm [10, pp. 140-143], which can be considered as a generalization of the Green's function mentioned above. Indeed, the Green's form $\gamma^\rho(z, \mathfrak{p})$ is defined as a solution of the inhomogeneous Laplace equation having the typical singularity at $\mathfrak{p}$, and, in the simplest case of compact Riemann

surfaces, the Green's form $\gamma^0(z, \mathfrak{p})$ of the rank 0 coincides with the Green's function $\gamma(z, \mathfrak{p})$ defined above. We may expect therefore that the Green's form $\gamma^{2n-2}(z, \mathfrak{p})$ of the rank $2n-2$ acts in $\mathfrak{M}$ just as the Green's function in the case of Riemann surfaces so that *the multiplicative meromorphic function with the divisor D is given by such a formula as* (1.1) *in terms of the integral*

$$\gamma[D](z) = \int_D \gamma^{2n-2}(z, \quad)$$

of $\gamma^{2n-2}(z, \mathfrak{p})$ extending over the divisor D, which corresponds to the sum $\gamma[\mathfrak{d}](z)$ defined in (1.2). According to this idea, we shall first examine, in §2, the properties of the integral $\gamma[Z](z)$ of the Green's form extending over an arbitrary cycle Z, and then, in §3, we shall deduce a formula representing the Picard integral of the third kind with the logarithmic polar cycle D in terms of the integral $\gamma[D](z)$. The formula representing the multiplicative meromorphic function with the given divisor D in terms of $\gamma[D](z)$ will be obtained in the last §4. Finally we shall prove a theorem concerning the necessary and sufficient condition for D in order that D is the divisor of a one-valued meromorphic function, which can be considered as a generalization of Abel's theorem in the classical theory of algebraic functions [13, pp. 126-127].

2. Green's forms. Let $\mathfrak{M}$ be a n-dimensional (topologically $2n$-dimensional) compact analytic variety with a positive definite Kählerian metric

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$$

where $z^1, z^2, \ldots, z^n$ denote local analytic coordinates on $\mathfrak{M}$. Putting

$$z^\alpha = x^\alpha + ix^{n+\alpha} \qquad (\alpha = 1, 2, \ldots, n),$$

we introduce the real coordinates $x^1, x^2, \ldots, x^{2n}$ on $\mathfrak{M}$; then $\mathfrak{M}$ becomes a $2n$-dimensional compact orientable Riemannian variety with the positive definite metric

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta = g_{jk}dx^j dx^k$$

(in what follows Latin subscripts j, k, etc., take values ranging from 1 to $2n$ and Greek subscripts α, β denote $1, 2, \ldots, n-1$ or n). Now we shall consider differential forms

$$\psi = \psi^\rho = \left(\frac{1}{\rho!}\right)\psi_{jk\ldots l}[dx^j dx^k \ldots dx^l]$$

defined on $\mathfrak{M}$, where ρ denotes the rank of ψ. A differential form of rank ρ will be called simply a *ρ-form*. A ρ-form ψ is said to be *measurable*, to *have continuous derivatives* or to be *regular*, if the coefficients $\psi_{jk\ldots l}$ are measurable, have continuous derivatives or are regular analytic functions of the real local coordinates $x^1, x^2, \ldots, x^{2n}$. We denote the derived form of ψ by $d\psi$ and the dual form of ψ by $*\psi$ or ψ^*; as is well known, they are defined respectively by

$$d\psi^\rho = \left(\frac{1}{\rho!}\right)[d\psi_{jk\ldots l}dx^j dx^k \ldots dx^l],$$

$$*\psi^\rho = \frac{g_{ip}g_{jq}\cdots g_{kr}}{g^{\frac{1}{2}}\rho!(2n-\rho)!}\ \operatorname{sgn}\begin{pmatrix} i\ j\ \ldots\ k\ l\ \ldots\ m \\ 1\ 2\ \ldots\ \ldots\ \ldots\ 2n \end{pmatrix}\psi^\rho{}_{l\ldots m}[dx^p dx^q \ldots dx^r],$$

where $g = |g_{jk}|$. The dual derivation δ and the Laplacian Δ are defined as[4]

$$\delta = *d*,$$

$$\Delta = -d\delta - \delta d.$$

Again, we introduce the "contraction product"

$$\psi^\rho \cdot \phi^\sigma = \phi^\sigma \cdot \psi^\rho = \left(\frac{1}{(\rho-\sigma)!\,\sigma!}\right)\psi_{ij\ldots kl\ldots m}\phi^{l\cdots m}[dx^i dx^j \ldots dx^k] \quad (\sigma \leqq \rho),$$

and the "inner product"

$$(\psi^\rho, \phi^\rho)_G = \int_G \psi^\rho \cdot \phi^\rho g^{\frac{1}{2}} dG \qquad (dG = dx^1 dx^2 \ldots dx^{2n}),$$

where G means an arbitrary subdomain of $\mathfrak{M}$; especially in case $G = \mathfrak{M}$, we write[5] (ψ, ϕ) for $(\psi, \phi)_{\mathfrak{M}}$. Then we have the Green's formula

$$(2.1) \qquad (d\phi^\rho, \psi^{\rho+1})_G - (\phi^\rho, \delta\psi^{\rho+1})_G = \int_B (\phi \cdot \psi)^j g^{\frac{1}{2}} do_j,$$

where B is the boundary of the domain G and do_j denotes the surface element $(-1)^{j-1}[dx^1 \ldots dx^{j-1}dx^{j+1} \ldots dx^{2n}]$. We introduce furthermore the "absolute value"

$$|\psi(\mathfrak{p})| = |\psi(\mathfrak{p}) \cdot \overline{\psi(\mathfrak{p})}|^{\frac{1}{2}} = \left|\left(\frac{1}{\rho!}\right)\psi_{jk\ldots l}(\mathfrak{p})\bar{\psi}^{jk\ldots l}(\mathfrak{p})\right|^{\frac{1}{2}}$$

of ψ at a point $\mathfrak{p}$ in $\mathfrak{M}$ and, by its means, define the norm $||\psi||_G$ as

$$||\psi||_G = \int_G |\psi(\mathfrak{p})|\, g^{\frac{1}{2}} dG_{\mathfrak{p}}.$$

In case $G = \mathfrak{M}$, we write $||\psi||$ for $||\psi||_{\mathfrak{M}}$. Incidentally, by a chain or a cycle will be meant a chain or a cycle with real coefficients; the boundary of a chain C will be denoted by ∂C. Again, if Z is a bounding cycle in $\mathfrak{M}$, we write $Z \sim 0$.

A form ψ is said to be *regular harmonic* in a subdomain G of $\mathfrak{M}$, if ψ is regular and satisfies the differential equations $d\psi = 0$, $\delta\psi = 0$ everywhere in G. By a *harmonic form* in $\mathfrak{M}$ we shall mean a form ψ which is regular harmonic in $\mathfrak{M}$ except for a nowhere dense compact subset S of $\mathfrak{M}$; then ψ is said to be *regular*

[4] Cf. de Rahm [10], Kodaira [5]. Our use of the notations d, δ coincides with that of de Rham [10], while our $*\psi^\rho$ $\Delta\psi^\rho$ correspond to $(-1)^\rho *\psi^\rho$, $-\Delta\psi^\rho$ of de Rham [10]. In Kodaira [5], we write $\mathfrak{r}^*$, $\mathfrak{r}$, $\mathfrak{d}$ for d, δ, $*$, respectively.

[5] In the present paper we do not use the well known "outer product".

in $\mathfrak{M} - S$ and to be *singular* on S. In case ψ is regular harmonic everywhere in $\mathfrak{M}$, ψ is called a *harmonic form of the first kind*. The linear space consisting of all real harmonic ρ-forms of the first kind will be denoted by $\mathfrak{E}^\rho$. Then $\mathfrak{E}^\rho$ constitutes a finite dimensional Euclidean vector space with respect to the inner product introduced above.[6] Choose a normalized orthogonal base $\{e_1^\rho, e_2^\rho, \ldots, e_b^\rho\}$ of the space $\mathfrak{E}^\rho$ arbitrarily and put

$$w^\rho(x, \xi) = \Sigma_\nu e_\nu^\rho(x) e_\nu^\rho(\xi).$$

The double harmonic ρ-form $w^\rho(x, \xi)$ thus defined is obviously independent of the choice of the base $\{e_\nu^\rho\}$ and therefore determined uniquely by $\mathfrak{M}$. Now we associate with every differentiable ρ-cycle Z on $\mathfrak{M}$ the harmonic ρ-form.

$$w[Z](x) = \int_Z w^\rho(x, \quad).$$

Then a famous theorem of W. V. D. Hodge can be stated as follows:

THEOREM 1 (HODGE).[7] *The mapping $Z \to w[Z]$ gives an isomorphism between the ρ-Betti group (over the real field) of $\mathfrak{M}$ and the space $\mathfrak{E}^\rho$ of all real harmonic ρ-forms of the first kind attached to $\mathfrak{M}$.*

In what follows this theorem will be cited as *Hodge's Theorem*. The character of the isomorphism $Z \to w[Z]$ will become more clear if we notice that the relation[8]

$$\int_\zeta w^*[Z] = I(\zeta, Z) \tag{2.2}$$

holds for an arbitrary $(2n - \rho)$-cycle ζ, where $I(\zeta, Z)$ means the intersection number of ζ and Z.

Now we put

$$w[\phi^\rho](\xi) = (w^\rho(\quad, \xi), \phi^\rho) = \Sigma_\nu(\phi^\rho, e_\nu^\rho) \cdot e_\nu^\rho(\xi)$$

for an arbitrary ρ-form ϕ^ρ with $(\phi^\rho, \bar{\phi}^\rho) < +\infty$. Then we have

THEOREM 2 (DE RHAM [10 pp. 140-143]). *For each ρ, $0 \leqq \rho \leqq 2n$, there exists on $\mathfrak{M}$ one and only one real double ρ-form*

$$\gamma^\rho(x, \xi) = \left(\frac{1}{\rho!}\right)^2 \gamma_{jk\ldots}{}_{\alpha\beta\ldots\nu}(x, \xi)[dx^j dx^k \ldots dx^l][d\xi^\alpha d\xi^\beta \ldots d\xi^\nu]$$

satisfying the following three conditions:

(i) *for every fixed ξ on $\mathfrak{M}$, $\gamma^\rho(x, \xi)$ is regular with respect to $x^1, x^2, \ldots, x^{2n}$ except for $x = \xi$ and satisfies*

$$\Delta_x \gamma^\rho(x, \xi) = w^\rho(x\ \xi); \tag{2.3}$$

[6] The fact that the space $\mathfrak{E}^\rho$ has a finite dimension can be proved independently of a famous theorem of Hodge. See de Rham [10, p. 138].

[7] Hodge [3, chap. III], Weyl [15]; see also Kodaira [5, §5].

[8] de Rham [10, p. 147]; see also Kodaira [5, p. 640, Theorem 17].

(ii) *in some neighbourhood $N(\xi)$ of ξ, $\gamma^\rho(x, \xi)$ is represented as*

$$\gamma^\rho(x, \xi) = \mathrm{E}^\rho(x, \xi) + \mu_\xi(x), \tag{2.4}$$

where μ_ξ is regular in $N(\xi)$, and $\mathrm{E}^\rho(x, \xi)$ means an elementary solution [5, §7] of Laplace's equation $\Delta \mathrm{E} = 0$ having typical singularities at ξ (therefore we have $\|\gamma^\rho(\ , \xi)\| < +\infty$);

(iii) *$\gamma^\rho(\ , \xi)$ is orthogonal to all harmonic ρ-forms of the first kind, i.e. we have*

$$(\gamma^\rho(\ , \xi), e) = 0, \qquad \text{for all } e \in \mathfrak{E}^\rho. \tag{2.5}$$

The double ρ-form $\gamma^\rho(x, \ \xi)$ is symmetric:

$$\gamma^\rho(x, \xi) = \gamma^\rho(\xi, x); \tag{2.6}$$

thus, for arbitrary fixed x, $\gamma^\rho(x, \ \xi)$ is regular with respect to $\xi^1, \ldots, \xi^{2n}$ except for $\xi = x$. We have the identities

$$\delta_x \gamma^\rho(x, \xi) = d_\xi \gamma^{\rho-1}(x, \xi), \tag{2.7}$$

$$*_x *_\xi \gamma^\rho(x, \xi) = \gamma^{2n-\rho}(x, \xi). \tag{2.8}$$

As a function of $4n$ variables $x^1\ x^2, \ldots, x^{2n}, \xi^1, \ldots, \xi^{2n}, \gamma^\rho(x, \xi)$ admits continuous derivatives of arbitrary orders except for $x = \xi$. Furthermore the norms $\|d\gamma^\rho(\ , \xi)\|$, $\|\delta\gamma^\rho(\ , \xi)\|$ of the derived forms $d_x\gamma^\rho(x, \xi)$, $\delta_x\gamma^\rho(x, \xi)$ are uniformly bounded with respect to ξ and, for an arbitrary form ψ^ρ with continuous first derivatives, the identity[9]

$$\psi^\rho(\xi) - w[\psi^\rho](\xi) = (d\gamma^\rho(\ , \xi), d\psi^\rho) + (\delta\gamma^\rho(\ , \xi), \delta\psi^\rho) \tag{2.9}$$

holds.

The double ρ-form $\gamma^\rho(x, \xi)$ is called the *Green's form* of rank ρ.

Incidentally, we shall mean by $\psi \subset G$ that the closure of the subset $\{\mathfrak{p};\ |\psi(\mathfrak{p})| \neq 0\}$ of $\mathfrak{M}$ is contained in G, G being an arbitrary subdomain of $\mathfrak{M}$. Then the principle of the method of orthogonal projections can be stated as follows:

THEOREM 3 (PRINCIPLE OF ORTHOGONAL PROJECTIONS).[10] *Let G be an open subset of $\mathfrak{M}$ and ψ be a measurable ρ-form defined in G with $\|\psi\|_G < +\infty$. Then, if ψ satisfies the integral equation*

$$(\psi, \Delta\eta)_G = 0$$

for all ρ-forms $\eta \subset G$ having continuous third derivatives, ψ is regular in G and satisfies $\Delta\psi = 0$. Again, if ψ satisfies the integral equations

$$(\psi, d\lambda)_G = 0, \qquad (\psi, \delta\eta)_G = 0$$

[9] This formula is an immediate consequence of the formula (4.5) in de Rham [10].

[10] Kodaira [5, pp. 608-609]. The method of orthogonal projections was first introduced by H. Weyl [14].

for arbitrary $(\rho-1)$- and $(\rho+1)$-forms λ, $\eta \subset G$ with continuous second derivatives, then ψ is regular harmonic in G.

Now we introduce for an arbitrary differentiable ρ-cycle Z, $0 \leqq \rho \leqq 2n-1$, the ρ-form

$$\gamma[Z](x) = \int_Z \gamma^\rho(x, \ ), \tag{2.10}$$

which will play a fundamental role in our theory. Then, denoting the support[11] of Z by $|Z|$, we have

THEOREM 4. *$\gamma[Z]$ is regular in $\mathfrak{M} - |Z|$ and satisfies the differential equations*

$$\delta\gamma[Z](x) = 0, \tag{2.11}$$

$$\delta d\gamma[Z](x) = -w[Z](x). \tag{2.12}$$

The derived form $d\gamma[Z]$ of $\gamma[Z]$ is regular harmonic in $\mathfrak{M} - |Z|$ if and only if Z is a bounding cycle on $\mathfrak{M}$. Furthermore $d\gamma[Z]$ has the finite norm: $\|d\gamma[Z]\| < +\infty$ and satisfies the integral equations

$$(d\gamma[Z], d\psi) = \int_Z \{\psi - w[\psi]\}, \tag{2.13}$$

$$(d\gamma[Z], \tau) = 0 \qquad (\delta\tau = 0), \tag{2.14}$$

where ψ means an arbitrary ρ-form having continuous first derivatives and τ an arbitrary $(\rho+1)$-form with continuous first derivatives satisfying $\delta\tau = 0$.

Proof. It is obvious that $\gamma[Z]$ is regular in $\mathfrak{M} - |Z|$. Now, using (2.7), we get

$$\delta\gamma[Z](x) = \int_Z \delta_z \gamma^\rho(x, \xi) = \int_Z d_\xi \gamma^{\rho-1}(x, \xi) = \int_{\partial Z} \gamma^{\rho-1}(x, \xi) = 0,$$

proving (2.11). Again, we obtain, using (2.3) and (2.11),

$$\begin{aligned} \delta d\gamma[Z](x) &= \int_Z \delta_x d_x \gamma^\rho(x, \xi) = -\int_Z \{d_x \delta_x \gamma^\rho(x, \xi) + w^\rho(x, \xi)\} \\ &= -d\delta\gamma[Z](x) - w[Z](x) = -w[Z](x), \end{aligned}$$

proving (2.12). Combined with the trivial relation $dd\gamma[Z](x) = 0$, (2.12) shows that $d\gamma[Z]$ is regular harmonic in $\mathfrak{M} - |Z|$ if and only if $Z \sim 0$ since, by virtue of Hodge's Theorem, $w[Z](x)$ vanishes identically if and only if $Z \sim 0$. The inequality $\|d\gamma[Z]\| < +\infty$ follows immediately from the fact that the norm $\|d\gamma^\rho(\ , \xi)\|$ is uniformly bounded with respect to ξ. Now, integrating the identity (2.9) over the cycle Z and using (2.11), we get immediately (2.13), while we have

$$(d\gamma[Z], \tau) = (\gamma[Z], \delta\tau) = 0,$$

proving (2.14).

[11] Z is a formal sum $\Sigma m_k T_k$ of a finite number of differentiable simplexes T_k lying in $\mathfrak{M}$ associated with real coefficients $m_k \neq 0$; then the support $|Z|$ is, by definition, the set theoretical sum ΣT_k of these simplexes.

In case Z is a bounding cycle: $Z = \partial C$, C being a differentiable $(\rho+1)$-chain in $\mathfrak{M}$, we have the formula [10, p. 143]

$$\int_\zeta {}^*d\gamma[\partial C] = I(\zeta, C) + (-1)^\rho \int_C w^*[\zeta], \tag{2.15}$$

where $I(\zeta, C)$ denotes the intersection number of ζ and C.

REMARK. The Green's forms $\gamma^\rho(x, \xi)$ are closely related to the double harmonic forms $e(x, \xi)$, $e^{**}(x, \xi)$ introduced in a recent paper of the author [5, §§14, 17]. Indeed, we have the relations

$$e(x, \xi) = -\delta_x d_x \gamma^\rho(x, \xi) = -\delta_x \delta_\xi \gamma^{\rho+1}(x, \xi),$$
$$e^{**}(x, \xi) = -d_x \delta_x \gamma^\rho(x, \xi) = -d_x d_\xi \gamma^{\rho-1}(x, \xi).$$

3. Picard integrals of the third kind. Now we introduce $2n$ formally independent variables $z^1, z^2, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n$ instead of $x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}$ and rewrite ρ-forms ψ as

$$\psi^\rho = \sum_{\sigma+\tau=\rho} \left(\frac{1}{\sigma!\tau!}\right) \psi_{\alpha\beta\ldots\gamma\dot{\delta}\ldots\dot{\varepsilon}} [\overbrace{dz^\alpha dz^\beta \ldots dz^\gamma}^{\sigma} \overbrace{d\bar{z}^\delta \ldots d\bar{z}^\varepsilon}^{\tau}].$$

Again, we introduce as usual the formal partial differentiation operators

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial x^{n+\alpha}}\right),$$
$$\frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial x^{n+\alpha}}\right).$$

As is well known, a function $f(x^1, x^2, \ldots, x^{2n})$ of real variables $x^1, x^2, \ldots, x^{2n}$ with continuous first derivatives is regular analytic with respect to complex variables $z^1, z^2, \ldots, z^n$ if and only if $\partial f/\partial \bar{z}^1, \partial f/\partial \bar{z}^2, \ldots, \partial f/\partial \bar{z}^n$ vanish identically. Incidentally, a ρ-form ψ of the type

$$\psi = \left(\frac{1}{\rho!}\right) \psi_{\alpha\beta\ldots\gamma} [dz^\alpha dz^\alpha \ldots dz^\gamma], \qquad (0 \leqslant \rho \leqslant n)$$

will be called *regular analytic* in a domain G, if the coefficients $\psi_{\alpha\beta\ldots\gamma}$ are regular analytic functions of *complex variables* $z^1, z^2, \ldots, z^n$ (whereas by a *regular* ρ-form we mean a ρ-form with coefficients which are regular analytic functions of *real coordinates* $x^1, x^2, \ldots, x^{2n}$).

Since, by hypothesis, the metric $ds^2 = 2g_{\alpha\dot{\beta}} dz^\alpha d\bar{z}^\beta$ is Kählerian, the 2-form

$$\omega = g_{\alpha\dot{\beta}} [dz^\alpha d\bar{z}^\beta]$$

is a harmonic form of the first kind [3, pp. 168-171]. Now we define two linear operators $\mathfrak{C}, \Lambda$ acting on differential forms as follows[12]: $\mathfrak{C}$ is the operator which

[12]These two operators were first introduced by W. V. D. Hodge [3, p. 171]. The simple definitions of $\mathfrak{C}, \Lambda$ employed here are due to Weil [12].

transforms dz^α, $d\bar{z}^\alpha$ into $i\,dz^\alpha$, $-i\,d\bar{z}^\alpha$, respectively; Λ is the operator which transforms ψ into $i\omega\cdot\psi$, where "$\cdot$" means the contraction product introduced in §2. In tensor notations, $\mathfrak{C}$, Λ are therefore defined as

$$(\mathfrak{C}\psi)_{\alpha\ldots\beta\,\dot{\gamma}\ldots\dot{\delta}}^{\overbrace{\quad}^{\sigma}\ \overbrace{\quad}^{\tau}} = i^{\sigma-\tau}\psi_{\alpha\ldots\beta\,\dot{\gamma}\ldots\dot{\delta}},$$

$$(\Lambda\psi)_{\alpha\ldots\beta\,\dot{\gamma}\ldots\dot{\delta}} = i\,g^{\lambda\dot{\rho}}\psi_{\lambda\dot{\rho}\alpha\ldots\beta\,\dot{\gamma}\ldots\dot{\delta}}.$$

Then we have

$$\Delta\mathfrak{C} = \mathfrak{C}\Delta, \tag{3.1}$$

$$\Delta\Lambda = \Lambda\Delta, \tag{3.2}$$

$$\delta\Lambda = \Lambda\delta, \tag{3.3}$$

$$\Lambda\mathfrak{C} = \mathfrak{C}\Lambda, \tag{3.4}$$

$$\Lambda d - d\Lambda = \mathfrak{C}^{-1}\delta\mathfrak{C}, \tag{3.5}$$

$$\mathfrak{C}\mathfrak{C}\psi^p = (-i)^p\psi^p, \tag{3.6}$$

and

$$\Lambda^{n-1}{}_*\psi^1 = -(-1)^{n(n+1)/2}(n-1)!\,\mathfrak{C}\psi^1. \tag{3.7}$$

The formulae (3.1), (3.2), (3.3), (3.6) were proved by W. V. D. Hodge [3, pp. 165-168, 171] by straightforward calculations, while (3.4) can be readily verified. The formula (3.5) is due to A. Weil [12]. The formula (3.7) can be proved also by a mere calculation, so the proof might be omitted. But, because of the importance of that formula, we shall give here a brief sketch of the calculation. By the definition of the operator Λ, we have

$$\begin{aligned}(\Lambda^{n-1}{}_*\psi^1)^\lambda &= (-i)^{n-1}\overbrace{g_{\alpha\dot{\imath}}g_{\beta\dot{\jmath}}\cdots g_{\gamma\dot{\sigma}}}^{n-1}(*\psi^1)^{\gamma\dot{\sigma}\cdots\beta\dot{\imath}\alpha\dot{\imath}\lambda}\\ &= (-1)^{(n-1)n/2}i^{n-1}g_{\alpha\dot{\imath}}g_{\beta\dot{\jmath}}\cdots g_{\gamma\dot{\sigma}}(*\psi^1)^{\lambda\alpha\beta\cdots\gamma\dot{\imath}\dot{\jmath}\cdots\dot{\sigma}}\\ &= (-1)^{(n-1)n/2}i^{n-1}(n-1)!\,\Sigma_\mu(-1)^{\mu-1}G_{\lambda\dot{\mu}}(*\psi^1)^{1\cdots n\dot{1}\cdots\dot{\mu}-1\,\dot{\mu}+1\cdots\dot{n}}\end{aligned}$$

$$\begin{aligned}(\Lambda^{n-1}{}_*\psi^1)^{\dot{\mu}} &= (-i)^{n-1}\overbrace{g_{\alpha\dot{\imath}}g_{\beta\dot{\jmath}}\cdots g_{\gamma\dot{\sigma}}}^{n-1}(*\psi^1)^{\gamma\dot{\sigma}\cdots\beta\dot{\imath}\alpha\dot{\mu}}\\ &= (-1)^{n(n-1)/2}(-i)^{n-1}g_{\alpha\dot{\imath}}g_{\beta\dot{\jmath}}\cdots g_{\gamma\dot{\sigma}}(*\psi^1)^{\alpha\beta\cdots\gamma\dot{\imath}\dot{\jmath}\cdots\dot{\sigma}}\\ &= (-1)^{n(n-1)/2}(-i)^{n-1}(n-1)!\,\Sigma_\lambda(-1)^{\lambda-1}G_{\lambda\dot{\mu}}(*\psi^1)^{1\cdots\lambda-1\,\lambda+1\cdots n\dot{1}\cdots\dot{n}},\end{aligned}$$

where these $G_{\lambda\dot{\mu}}$ mean minor determinants $|g_{\alpha\dot{\beta}}|$, while ${}_*\psi^1$ is given by

$$\begin{cases}(*\psi^1)^{1\cdots n\dot{1}\cdots\dot{\mu}-1\,\dot{\mu}+1\cdots\dot{n}} = g^{-1}(-1)^{1+\mu}(2/i)^n\psi^1_{\dot{\mu}},\\ (*\psi^1)^{1\cdots\lambda-1\,\lambda+1\cdots n\dot{1}\cdots\dot{n}} = g^{-1}(-1)^{\lambda+n-1}(2/i)^n\psi^1{}_\lambda.\end{cases}$$

Using the relations

$$\Sigma_\lambda g_{\lambda\dot{\nu}}G_{\lambda\dot{\nu}} = \delta_{\mu\dot{\nu}}|g_{\alpha\dot{\beta}}|,\quad \Sigma_\mu g_{\lambda\dot{\mu}}G_{\nu\dot{\mu}} = \delta_{\lambda\nu}|g_{\alpha\dot{\beta}}|,\quad g^{\frac{1}{2}} = 2^n|g_{\alpha\dot{\beta}}|,$$

we get therefore

$$(\Lambda^{n-1}{}_{*}\psi^{1})_{\bar{\mu}} = (-1)^{n(n+1)/2}(n-1)!\, i\, \psi^{1}{}_{\bar{\mu}},$$

$$(\Lambda^{n-1}{}_{*}\psi^{1})_{\lambda} = -(-1)^{n(n+1)/2}(n-1)!\, i\, \psi^{1}{}_{\lambda},$$

proving (3.7). Furthermore we have

$$\overline{\mathfrak{C}\,\psi} = \mathfrak{C}\,\bar{\psi}, \tag{3.8}$$

$$(\mathfrak{C}\,\phi,\ \overline{\mathfrak{C}\,\psi}) = (\phi, \bar{\psi}). \tag{3.9}$$

As is well known, from the formula

$$-(\Delta\psi, \bar{\psi}) = (d\psi, \overline{d\psi}) + (\delta\psi, \overline{\delta\psi})$$

it follows that *a form ψ with continuous second derivatives is a harmonic form of the first kind if and only if ψ satisfies $\Delta\psi = 0$ everywhere in $\mathfrak{M}$.* Combined with this fact, (3.1) shows that, if e is a harmonic form of the first kind, $\mathfrak{C}e$ is also a harmonic form of the first kind. Whence we conclude, using (3.8) and (3.9), that $\mathfrak{C}e_1{}^{\rho}, \mathfrak{C}e_2{}^{\rho}, \ldots, \mathfrak{C}e_b{}^{\rho}$ *constitute a normalized orthogonal base of the space* $\mathfrak{E}^{\rho}$, where $\{e_1{}^{\rho}, e_2{}^{\rho}, \ldots, e_b{}^{\rho}\}$ is a normalized orthogonal base of $\mathfrak{E}^{\rho}$ introduced in §2. We get therefore

$$\Sigma_{\nu}\mathfrak{C}e_{\nu}{}^{\rho}(x)\mathfrak{C}e_{\nu}{}^{\rho}(\xi) = \Sigma_{\nu}e_{\nu}{}^{\rho}(x)e_{\nu}{}^{\rho}(\xi),$$

or

$$\mathfrak{C}_x\mathfrak{C}_\xi w^{\rho}(x,\xi) = w^{\rho}(x,\xi). \tag{3.10}$$

LEMMA. *We have*

$$\mathfrak{C}_x\mathfrak{C}_\xi\gamma^{\rho}(x,\xi) = \gamma^{\rho}(x,\xi). \tag{3.11}$$

Proof. Put, for simplicity's sake, $\tilde{\gamma}(x,\xi) = \mathfrak{C}_x\mathfrak{C}_\xi\gamma^{\rho}(x,\xi)$, $\tilde{\mathrm{E}}(x,\xi) = \mathfrak{C}_x\mathfrak{C}_\xi \mathrm{E}(x,\xi)$. We fix the point ξ and consider γ^{ρ}, $\tilde{\gamma}$, E^{ρ}, $\tilde{\mathrm{E}}$ as functions of x. The "highest term" of the elementary solution $\mathrm{E}^{\rho}(x,\xi)$ is given by

$$[(2n-2)\,\Omega_{2n}]^{-1} r(x,\xi)^{2-2n} \times$$

$$\sum_{\sigma+\tau=\rho}\left(\frac{1}{\sigma!\tau!}\right)\underbrace{g_{\alpha\bar{\lambda}}\cdots g_{\beta\bar{\lambda}}}_{\sigma}\ \underbrace{g_{\mu\bar{\gamma}}\cdots g_{\nu\bar{\delta}}}_{\tau}\,[dz^{\alpha}\ldots dz^{\beta}\,d\bar{z}^{\gamma}\ldots d\bar{z}^{\delta}][d\bar{\zeta}^{\varepsilon}\ldots d\bar{\zeta}^{\lambda}d\zeta^{\mu}\ldots d\zeta^{\nu}],$$

where $g_{\alpha\bar{\lambda}} = g_{\alpha\bar{\lambda}}(\xi)$, $z^{\alpha} = x^{\alpha} + ix^{n+\alpha}$, $\zeta^{\lambda} = \xi^{\lambda} + i\xi^{n+\lambda}$, Ω_{2n} is the surface area of a $2n$-dimensional unit sphere, and $r(x,\xi)$ means the geodesic distance from x to ξ [5 §7]. It is obvious that this highest term remains unchanged by simultaneous application of the operators $\mathfrak{C}_x$, $\mathfrak{C}_\xi$. Hence we have

$$\tilde{\mathrm{E}}(x,\xi) - \mathrm{E}^{\rho}(x,\xi) = r^{2-2n}O(r) + \log(1/r).\, O(1) \qquad (r = r(x,\xi)),$$

where $O(r)$, $O(1)$ mean holomorphic functions of $x^1 - \xi^1, \ldots, x^{2n} - \xi^{2n}$ having the orders $O(r)$, $O(1)$, respectively. On the other hand, it is obvious by (3.1) that $\tilde{\mathrm{E}}(x,\xi)$ satisfies also $\Delta\tilde{\mathrm{E}}(x,\xi) = 0$. Let $N(\xi)$ be a sufficiently small neighbourhood of ξ. Then we have therefore

$$(\tilde{\mathbf{E}}(\ ,\xi) - \mathbf{E}^{\rho}(\ ,\xi), \Delta\eta)_{N(\xi)} = (\Delta\tilde{\mathbf{E}} - \Delta\mathbf{E}, \eta)_{N(\xi)} = 0$$

for an arbitrary ρ-form $\eta \subset N(\xi)$ with continuous second derivatives, and

$$\| \tilde{\mathbf{E}}(\ ,\xi) - \mathbf{E}^{\rho}(\ ,\xi)\|_{N(\xi)} < +\infty.$$

Hence, by virtue of the principle of orthogonal projections, $\tilde{\mathbf{E}}(\ ,\xi) - \mathbf{E}^{\rho}(\ ,\xi)$ is regular everywhere in $\mathrm{N}(\xi)$. Now, using (3.1) and (3.10) we get from (2.3)

$$\Delta_x \tilde{\gamma}(x, \xi) = w^{\rho}(x, \xi), \qquad (x \neq \xi) \tag{3.12}$$

while from (2.4) follows

$$\tilde{\gamma}(x, \xi) = \tilde{\mathbf{E}}(x, \xi) + \tilde{\mu}(x), \qquad \tilde{\mu} \text{ is regular in } \mathrm{N}(\xi). \tag{3.13}$$

Comparing (3.12), (3.13) with (2.3), (2.4), we infer from the above result that $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is regular everywhere in $\mathfrak{M}$ and satisfies

$$\Delta\{\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)\} = 0;$$

hence $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is a harmonic form of the first kind. On the other hand, from (2.5) and (3.9) follows

$$(\tilde{\gamma}(\ ,\xi), \mathfrak{C}e) = 0, \qquad \text{for all } e \in \mathfrak{E}^{\rho},$$

while the mapping $e \to \mathfrak{C}e$ maps $\mathfrak{E}^{\rho}$ isometrically *on* itself. Hence $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi)$ is orthogonal to all harmonic ρ-forms of the first kind and therefore $\tilde{\gamma}(\ ,\xi) - \gamma^{\rho}(\ ,\xi) = 0$, q.e.d.

A compact subset Γ of $\mathfrak{M}$ will be called a *closed analytic surface*, if, for every point $\mathfrak{p} \in \mathfrak{M}$, there exists a regular analytic function $f_{\mathfrak{p}}(z)$ of complex coordinates $z^1, z^2, \ldots, z^n$ defined in a neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ such that Γ coincides in $N(\mathfrak{p})$ with the zero-point variety of $f_{\mathfrak{p}}(z)$; then

$$f_{\mathfrak{p}}(z) = 0$$

will be called a local equation of Γ at $\mathfrak{p}$. Choose the local coordinates $z^1, z^2, \ldots, z^n$ so that $\mathfrak{p}$ coincides with the origin $(0, 0, \ldots, 0)$. Then the set of all regular analytic functions

$$h(z) = c_0 + \Sigma c_{\alpha} z^{\alpha} + \frac{1}{2!}\Sigma c_{\alpha\beta} z^{\alpha} z^{\beta} + \frac{1}{3!}\Sigma c_{\alpha\beta\gamma} z^{\alpha} z^{\beta} z^{\gamma} + \ldots$$

defined in some (not fixed) neighbourhoods of $\mathfrak{p} = (0, 0, \ldots, 0)$ constitutes a ring $\mathfrak{o}_{\mathfrak{p}}$ without null divisor, in which every $h(z)$ with $h(0) \neq 0$ is considered as a unit. As an element of $\mathfrak{o}_{\mathfrak{p}}$, $f_{\mathfrak{p}}(z)$ can be decomposed into the power product

$$f_{\mathfrak{p}}(z) = U(z) \cdot \Pi_j \{f_{\mathfrak{p}}^{(j)}(z)\}^{m_j} \qquad (U(0) \neq 0)$$

of irreducible factors $f_{\mathfrak{p}}^{(j)}(z)$, where the decomposition is unique up to the unit[13] $U(z)$. In accordance with this Γ is decomposed in a neighbourhood of $\mathfrak{p}$ into the sum

[13] Cf. Bochner and Martin [1, chap. IX].

$$\Gamma'_{\mathfrak{p}} \cup \Gamma''_{\mathfrak{p}} \cup \ldots \cup \Gamma_{\mathfrak{p}}^{(j)} \cup \ldots$$

of the *branches* $\Gamma_{\mathfrak{p}}^{(j)}$, each of which is the zero-point variety of the corresponding factor $f_{\mathfrak{p}}^{(j)}(z)$. Obviously the local equation $f_{\mathfrak{p}}(z) = 0$ of Γ at $\mathfrak{p}$ is equivalent to

$$f_{\mathfrak{p}}^{*}(z) = \Pi_j f_{\mathfrak{p}}^{(j)}(z) = 0,$$

which will be called a *minimal local equation* of Γ at $\mathfrak{p}$. The minimal local equation is characterized by the following property: *If $f_{\mathfrak{p}}^{*}(z)$ is represented as the product*

$$f_{\mathfrak{p}}^{*}(z) = q(z)h(z)$$

of two functions in $\mathfrak{o}_{\mathfrak{p}}$ and $q(z)$ vanishes on Γ in some neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, then $h(z)$ is a unit in $\mathfrak{o}_{\mathfrak{p}}$, i.e., $h(0) \neq 0$. In case, in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, Γ consists of a single branch $\Gamma'_{\mathfrak{p}}$ and (at least) one of the partial derivatives $\partial f'_{\mathfrak{p}}/\partial z^{\alpha}$ $(\alpha = 1, 2, \ldots, n)$ does not vanish at $\mathfrak{p}$, then $\mathfrak{p}$ is called a *simple point* of Γ; otherwise $\mathfrak{p}$ is a *singular point. If $\mathfrak{p}$ is a simple point of Γ, we can choose the system of local coordinates with the origin $\mathfrak{p}$ so that the minimal local equation of Γ at $\mathfrak{p}$ becomes the simple form $z^1 = 0$.* The set S of all singular points of Γ is an analytic subvariety of Γ of the complex dimension $\leqq n - 2$, which is called the *singular locus* of Γ. In case $\Gamma - S$ is a connected set, Γ is called *irreducible*; otherwise Γ is said to be *reducible*. A reducible closed analytic surface can be decomposed uniquely into the sum of a finite number of irreducible ones. From the topological viewpoint, an irreducible closed analytic surface Γ is an orientable $(2n-2)$-*pseudo-manifold* with respect to the "natural" orientation induced by local analytic parameter systems on $\Gamma - S$, and thus Γ *is a* $(2n-2)$-*cycle.*

Now, suppose a bounding $(2n-2)$-*cycle* $D = \Sigma m_k \Gamma_k$ on $\mathfrak{M}$ with real coefficients m_k consisting of a finite number of irreducible closed analytic surfaces Γ_k as given and consider the $(2n-2)$-form

$$\gamma[D](x) = \int_D \gamma^{2n-2}(x, \ ) \tag{3.14}$$

associated with D. It is to be noted here that the cycle D is not necessarily differentiable in the usual sense. We assume, as usual, that *the variety $\mathfrak{M}$ is triangulated with its subvarieties Γ_k into "analytic simplexes"* so that $\mathfrak{M}$ becomes a finite simplicial complex K containing Γ_k as its subcomplexes.[14] Then, on each ρ-simplex T of K, we can choose a real parameter system $\{t^1, t^2, \ldots, t^{\rho}\}$ describing every point $\mathfrak{p}$ on T as $\mathfrak{p} = \mathfrak{p}(t^1, t^2, \ldots, t^{\rho})$ so that the coordinates $x^j(\mathfrak{p}(t))$ of $\mathfrak{p}(t)$ have continuous first derivatives with respect to $t^1, t^2, \ldots, t^{\rho}$ in every *inner* point $\mathfrak{p}$ of T, but the differentiability of $x^j(\mathfrak{p}(t))$ might break down on the *boundary* of T; while, in the usual definition of the "differentiable simplex," we request the existence of a parameter system $\{t^1, \ldots, t^{\rho}\}$ such that $x^j(p(t))$ admit continuous derivatives with respect to $t^1, \ldots t^{\rho}$ everywhere

[14] As to the possibility of triangulation, see Koopman and Brown [7], Lefschetz and Whitehead [9]; cf. also van der Waerden [11], Lefschetz [8, pp. 362-369].

in the simplex *including the boundary*. Thus, the cycle D might not be differentiable. But, as one readily infers, each ρ-simplex T of K has a finite (ρ-dimensional) area, and therefore *the integral*

$$\int_T \phi^\rho = \frac{1}{\rho!} \int_T \phi_{j_1 \ldots j_\rho}(x) \frac{\partial(x^{j_1} x^{j_2} \ldots x^{j_\rho})}{\partial(t^1 t^2 \ldots t^\rho)} dt^1 dt^2 \ldots dt^\rho$$

converges absolutely for an arbitrary continuous ρ-form ϕ defined in a neighbourhood of T. This asserts the absolute convergence of the integral in (3.14). Furthermore, if we take for granted, as usual, the validity of the Green-Stokes' formula

$$\int_C d\psi = \int_{\partial C} \psi$$

for an arbitrary chain C of K, the arguments expounded in §2 can be applied to arbitrary cycles of K, and thus Theorem 4 is valid also for $\gamma[D]$. By virtue of Theorem 4, $d\gamma[D]$ is a real harmonic $(2n-1)$-form. Hence, putting

$$4\pi i \, {}_*d\gamma[D](x) = \phi_\alpha dz^\alpha - \bar{\phi}_\alpha d\bar{z}^\alpha \tag{3.15}$$

we can introduce a 1-form $\phi_\alpha dz^\alpha$. Now we shall prove that $\int \phi_\alpha dz^\alpha$ *is the Picard integral of the third kind having D as its "logarithmic polar cycle."*[15] Since D consists of analytic surfaces, the "surface elements" of the types

$$[dz^1 dz^2 \ldots dz^n \overbrace{d\bar{z}^\lambda \ldots d\bar{z}^\nu}^{n-2}],$$

$$[\overbrace{dz^\beta \ldots dz^\gamma}^{n-2} d\bar{z}^1 d\bar{z}^2 \ldots d\bar{z}^n]$$

vanish identically on D; consequently $\gamma[D](x)$ has the form

$$\gamma[D](x) = \frac{1}{(2n-2)!} \int_D \gamma_{\alpha\beta \ldots \gamma\lambda\mu \ldots \nu}(x, \xi) [d\xi^\alpha \overbrace{d\xi^\beta \ldots d\xi^\gamma}^{n-1} \overbrace{d\bar{\xi}^\lambda d\bar{\xi}^\mu \ldots d\bar{\xi}^\nu}^{n-1}],$$

where $\zeta^\alpha = \xi^\alpha + i\,\xi^{n+\alpha}$. Hence we get, using (3.6) and (3.11),

$$\mathbb{G}\gamma[D](x) = \int_D \mathbb{G}_x \gamma(x, \xi) = \int_D \mathbb{G}_x \mathbb{G}_x \mathbb{G}_\xi \gamma(x, \xi) = \int_D \mathbb{G}_\xi \gamma(x, \xi) = \int_D \gamma(x, \xi),$$

or

$$\mathbb{G}\gamma[D](x) = \gamma[D](x). \tag{3.16}$$

Keeping the relations $\mathbb{G}\Lambda = \Lambda\mathbb{G}$, $\delta\Lambda = \Lambda\delta$ in mind, we can readily deduce from (3.16) and (3.5) the formula

$$\Lambda^m d\gamma[D](x) = d\Lambda^m \gamma[D](x) \quad (m = 1, 2, \ldots, n-1). \tag{3.17}$$

Indeed, if one assumes that (3.17) were already proved for $m = k$, then we get

[15] Cf. Weil [12].

$$\Lambda^{k+1}d\gamma[\mathrm{D}] - d\Lambda^{k+1}\gamma[D] = (\Lambda d - d\Lambda)\Lambda^k\gamma[D]$$

$$= \mathfrak{C}^{-1}\delta\mathfrak{C}\Lambda^k\gamma[D] = \mathfrak{C}^{-1}\Lambda^k\delta\mathfrak{C}\gamma[D] = \mathfrak{C}^{-1}\Lambda^k\delta\gamma[D] = 0,$$

since, by (2.11), $\delta\gamma[D](x) = 0$. Thus (3.17) is proved by induction on m.

Now, using (3.7) and (3.17), we get from (3.15)

$$\phi_\alpha dz^\alpha + \bar{\phi}_\alpha d\bar{z}^\alpha = d\left\{\frac{4\pi(-1)^{n(n+1)/2}}{(n-1)!}\,\Lambda^{n-1}\gamma[D](x)\right\};$$

hence we obtain

$$\phi_\alpha dz^\alpha = d\left\{\frac{2\pi(-1)^{n(n+1)/2}}{(n-1)!}\,\Lambda^{n-1}\gamma[D](x)\right\} + 2\pi i\,{}_*d\gamma[D](x). \tag{3.18}$$

From (3.18) follows first that the coefficients ϕ_α are regular with respect to $x^1, x^2, \ldots, x^{2n}$ in $\mathfrak{M} - |D|$, where $|D|$ means the support of D, i.e. $|D| = \cup\,\Gamma_k$. Again, since $d\gamma[D](x)$ is regular harmonic in $\mathfrak{M} - |D|$, (3.18) yields immediately

$$d(\phi_\alpha dz_\alpha) = 0, \tag{3.19}$$

which implies $\partial\phi^\alpha/\partial\bar{z}^\beta = 0$ ($\alpha, \beta = 1, 2, \ldots, n$). Hence ϕ_α *are regular analytic with respect to complex coordinates* $z^1, z^2, \ldots, z^n$ in $\mathfrak{M} - |D|$. Moreover (3.19) shows that the integral $\int^z \phi_\alpha dz^\alpha$ is locally univalent in $\mathfrak{M} - |D|$. Thus, *putting*

$$\Phi(z) = \int^z \phi_\alpha dz^\alpha + \text{const.},$$

we obtain a many valued analytic function $\Phi(z)$ *on* $\mathfrak{M}$ *which is regular in* $\mathfrak{M} - |D|$. Furthermore, since, as one readily infers, the formula

$$\overline{\Lambda\psi} = \Lambda\bar{\psi}$$

holds, the 0-form $\Lambda^{n-1}\gamma[D](x)$ is real and therefore *the real part of* $\Phi(z)$ *is given by*

$$\Re\Phi(z) = 2\pi\{(-1)^{n(n+1)/2}/(n-1)!\}\Lambda^{n-1}\gamma[D](x) + \text{const.}$$

Thus $\Re\Phi(z)$ *is one-valued on* $\mathfrak{M}$.

Our next task is to determine the singularities of $\Phi(z)$ on $|D|$. For that purpose, denoting the minimal local equation of each irreducible component Γ_k of $|D|$ at $\mathfrak{p} \in |D|$ by $f_{k\mathfrak{p}}(z) = 0$, we shall compare $\Phi(z)$ with $\Sigma_k m_k \log f_{k\mathfrak{p}}(z)$ in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ and show that *the difference*

$$\Phi(z) - \Sigma_k m_k \log f_{k\mathfrak{p}}(z)$$

is regular analytic in $N(\mathfrak{p})$. Assume first $\mathfrak{p}$ to be a simple point of[16] $|D|$. Then, in $N(\mathfrak{p})$, $|D|$ consists of a single component, say Γ_1, and, after a suitable choice of the local coordinates with the origin $\mathfrak{p}$, the minimal local equation of Γ_1 at $\mathfrak{p}$ has the simple form

[16] Remember that $|D|$ is a closed analytic surface consisting of irreducible components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$.

$$f_{1\mathfrak{p}}(z) = z^1 = 0.$$

Now it can be readily verified that the 1-form $\Im d \log f_{1\mathfrak{p}}$ satisfies the integral equations

$$(\Im d \log f_{1\mathfrak{p}},\ \delta\eta)_N = 2\pi \int_{\Gamma_1} \eta^*, \tag{3.20}$$

$$(\Im d \log f_{1\mathfrak{p}},\ d\lambda)_N = 0, \tag{3.21}$$

where $N = N(\mathfrak{p})$ and η or λ means respectively an arbitrary 2- or 0-form $\subset N(\mathfrak{p})$ having continuous first derivatives. In fact, denoting by $G(\epsilon)$ the cylindrical domain $|z^1| < \epsilon$, by $T(\epsilon)$ the cylindrical surface $|z^1| = \epsilon$, and putting

$$z^1 = x^1 + i\, x^{n+1} = qe^{i\theta} \qquad (q = |z^1|),$$

we have

$$\Im\, d \log f_{1\mathfrak{p}} = d\theta = q^{-2}(x^1 dx^{n+1} - x^{n+1} dx^1)$$

and therefore, by virtue of the Green's formula (2.1),

$$(\Im d \log f_{1\mathfrak{p}},\ \delta\eta)_N = \lim_{\epsilon \to 0} (d\theta,\ \delta\eta)_{N-G(\epsilon)} = \lim_{\epsilon \to 0} \int_{T(\epsilon)} (d\theta \cdot \eta)^j g^{\frac{1}{2}} do_j$$

$$= \lim_{\epsilon \to 0} \int_{T(\epsilon)} q^{-2}(x^1 \eta^{j\ n+1} - x^{n+1}\eta^{j1}) g^{\frac{1}{2}} do_j = \lim_{\epsilon \to 0} \int_{T(\epsilon)} q^{-2} \eta^{1\ n+1} g^{\frac{1}{2}} (x^1 do_1 + x^{n+1} do_{n+1}),$$

since $\int_{T(\epsilon)} x^j do_k$ vanishes for $j \neq k$. Whence we get

$$(\Im d \log f_{1\mathfrak{p}},\ \delta\eta)_N = \lim_{\epsilon \to 0} (-1)^{n-1} \int_{T(\epsilon)} \eta^{1\ n+1} g^{\frac{1}{2}} [d\theta dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}]$$

$$= (-1)^{n-1} 2\pi \int_{\Gamma_1} \eta^{1\ n+1} g^{\frac{1}{2}} [dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}] = 2\pi \int_{\Gamma_1} \eta^*,$$

proving (3.20). Again we have

$$(\Im d \log f_{1\mathfrak{p}},\ d\lambda)_N = \lim_{\epsilon \to 0} (d\theta,\ d\lambda)_{N-G(\epsilon)} = -\lim_{\epsilon \to 0} \int_{T(\epsilon)} (\lambda d\theta)^j g^{\frac{1}{2}} do_j$$

$$= \lim_{\epsilon \to 0} \int_{T(\epsilon)} q^{-2}(x^{n+1} g^{j1} - x^1 g^{j\ n+1}) \lambda g^{\frac{1}{2}} do_j = \lim_{\epsilon \to 0} \int_{T(\epsilon)} q^{-2} g^{1\ n+1} \lambda g^{\frac{1}{2}} (x^{n+1} do_{n+1} - x^1 do_1)$$

$$= \lim_{\epsilon \to 0} (-1)^n \int_{T(\epsilon)} g^{1\ n+1} \lambda g^{\frac{1}{2}} (\cos^2\theta - \sin^2\theta)[d\theta dx^2 \ldots dx^n dx^{n+2} \ldots dx^{2n}]$$

proving (3.21). On the other hand, we infer from (2.13) and (2.14) that the 1-form $*d\gamma[D]$ satisfies

$$(*d\gamma[D],\ \delta\eta) = \int_D \eta^* = m_1 \int_{\Gamma_1} \eta^*, \qquad (*d\gamma[D],\ d\lambda) = 0.$$

Combined with (3.20), (3.21), these formulae yield

$$(2\pi_* d\gamma[D] - \Im m_1 d \log f_{1\mathfrak{p}},\ \delta\eta)_N = 0,$$
$$(2\pi_* d\gamma[D] - \Im m_1 d \log f_{1\mathfrak{p}},\ d\lambda)_N = 0,$$

while we have

$$\| 2\pi_* d\gamma[D] - \Im m_1 d \log f_{1\mathfrak{p}} \|_N \leqslant 2\pi \| d\gamma[D] \| + |m_1| \| d\theta \|_N < +\infty.$$

Hence by virtue of the principle of orthogonal projections, the difference

$$2\pi_* d\gamma[D] - \Im m_1 d \log f_{1\mathfrak{p}} = \Im \{ \phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}} \}$$

is regular harmonic in $N = N(\mathfrak{p})$. This implies that the difference $\phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}}$ is regular with respect to *real* coordinates $x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^{2n}$ everywhere in $N(\mathfrak{p})$, while, in $N(\mathfrak{p}) - \Gamma_1$, $\phi_\alpha dz^\alpha$ and $d \log f_{1\mathfrak{p}}$ are regular analytic with respect to *complex* coordinates $z^1, z^2, \ldots, z^n$. Hence $\phi_\alpha dz^\alpha - m_1 d \log f_{1\mathfrak{p}}$ is regular analytic with respect to complex variables $z^1, z^2, \ldots, z^n$ everywhere in $N(\mathfrak{p})$, and therefore *the difference*

$$\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$$

is regular analytic with respect to complex coordinates $z^1, \ldots, z^n$ *everywhere in* $N(\mathfrak{p})$, since, by hypothesis, $f_{k\mathfrak{p}}(z)$ $(k = 2, 3, \ldots, \kappa)$ do not vanish in $N(\mathfrak{p})$.

Consider now the case that $\mathfrak{p}$ is a singular point of $|D|$. Denoting the singular locus of $|D|$ by S, we infer from the above result that $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic with respect to complex coordinates $z^1, \ldots, z^n$ in every point $\mathfrak{q} \in N(\mathfrak{p}) - S$, $N(\mathfrak{p})$ being a sufficiently small neighbourhood of $\mathfrak{p}$ such that the local equations $f_{k\mathfrak{p}}(z) = 0$ of Γ_k are available in the whole of $N(\mathfrak{p})$. Indeed, denoting for each k the minimal local equation of Γ_k at $\mathfrak{q}$ by $f_{k\mathfrak{p}}(z) = 0$, $f_{k\mathfrak{q}}(z)$ can be written, in a sufficiently small neighbourhood $N(\mathfrak{q})$ of $\mathfrak{q}$, as

$$f_{k\mathfrak{p}}(z) = U_k(z) f_{k\mathfrak{q}}(z),$$

where $U_k(z)$ is a non-vanishing regular analytic function in $N(\mathfrak{q})$; hence we get

$$\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}} = \phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{q}} - \Sigma m_k d \log U_k$$

proving that $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic in $N(\mathfrak{q})$. Thus $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic in $N(\mathfrak{p})$ excepting at most the analytic subvariety $S \cap N(\mathfrak{p})$ of the complex dimension $\leqslant n - 2$, while, by a theorem of Hartogs [2], an analytic variety containing all singular points of an analytic function of n complex variables must have the complex dimension $n-1$. Hence $\phi_\alpha dz^\alpha - \Sigma m_k d \log f_{k\mathfrak{p}}$ is regular analytic everywhere in $N(\mathfrak{p})$. Thus we see that, for every point $\mathfrak{p} \in |D|$, *the integral* $\Phi(z) = \int^z \phi_\alpha dz^\alpha$ *can be written as*

$$\Phi(z) = \Sigma_k m_k \log f_{k\mathfrak{p}}(z) + \textit{regular part, in } N(\mathfrak{p}).$$

Finally we shall evaluate the period

$$[\Phi]_\Gamma = \int_\Gamma \phi_\alpha dz^\alpha$$

of the integral $\Phi(z)$ on an arbitrary 1-cycle ζ. Denote by C a $(2n-1)$-chain such that $\partial C = D$. Then, using (2.15), we get from (3.18) the formula

$$[\Phi]_\zeta = 2\pi i \{I(\zeta, C) + \int_C w^*[\zeta]\}. \tag{3.22}$$

Thus we have proved the following

THEOREM 5. *Let $D = \Sigma_k m_k \Gamma_k$ be a bounding $(2n-2)$-cycle on $\mathfrak{M}$ consisting of a finite number of closed analytic surfaces Γ_k with minimal local equations $f_{k\mathfrak{p}}(z) = 0$ $(k = 1, 2, \ldots, \kappa)$. Then the integral*

$$\Phi_D(z) = \frac{2\pi(-1)^{n(n+1)/2}}{(n-1)!} \Lambda^{n-1}\gamma[D](x) + 2\pi i \int_*^z d\gamma[D](x) + \text{const.}$$

is the Picard integral of the third kind with the logarithmic polar cycle D, i.e. $\Phi_D(z)$ is a many valued analytic function on $\mathfrak{M}$ which is regular in $\mathfrak{M} - |D|$ and, for every $\mathfrak{p} \in |D|$, $\Phi_D(z)$ has, in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$, the form

$$\Phi_D(z) = \Sigma m_k \log f_{k\mathfrak{p}}(z) + \textit{regular part},$$

where $\gamma[D](x)$ means the integral of the Green's form $\gamma^{2n-2}(x, \xi)$ over the cycle D:

$$\gamma[D](x) = \int_D \gamma^{2n-2}(x, \ ).$$

The real part

$$\Re\Phi_D(z) = 2\pi \{(-1)^{n(n+1)/2}/(n-1)!\} \Lambda^{n-1}\gamma[D](x)$$

of $\Phi_D(z)$ is one-valued on $\mathfrak{M}$. The period $[\Phi_D]_\zeta$ of the integral $\Phi_D(z)$ over an arbitrary 1-cycle ζ is given by

$$[\Phi_D]_\zeta = 2\pi i \{I(\zeta, C) + \int_C w^*[\zeta]\} \qquad (\partial C = D),$$

where C is a $(2n-1)$-chain with $\partial C = D$ and $w[\zeta]$ means the harmonic 1-form of the first kind associated with the 1-cycle ζ in the sense of Hodge's Theorem.

4. Multiplicative meromorphic functions. Now it is obvious how to construct a multiplicative meromorphic function having the given divisor D. Assume that a bounding $(2n-2)$-cycle $D = \Sigma m_k \Gamma_k$ on $\mathfrak{M}$ with *integral* coefficients m_k consisting of a finite number of closed analytic surfaces Γ_k is given. Then, constructing the Picard integral $\Phi_D(z)$ of the third kind as above and putting

$$F_D(z) = \exp\{\Phi_D(z)\},$$

we get the multiplicative meromorphic function $F_D(z)$ having D as its divisor. Thus we obtain the following

THEOREM 6. *Let $D = \Sigma m_k \Gamma_k$ be a bounding $(2n-2)$-cycle on $\mathfrak{M}$ with integral coefficients m_k consisting of a finite number of closed analytic surfaces Γ_k with minimal local equations $f_{k\mathfrak{p}}(z) = 0$ $(k = 1, 2, \ldots, \kappa)$. Then*

$$F_D(z) = c.\exp 2\pi\left\{\frac{(-1)^{n(n+1)/2}}{(n-1)!}\,\Lambda^{n-1}\gamma[D](x)+i\int^{z}{}_{x}d\gamma[D](x)\right\} \qquad (c \neq 0)$$

is a meromorphic function having D as its divisor, i.e. $F_D(z)$ is a many valued analytic function on $\mathfrak{M}$ which is regular in $\mathfrak{M} - |D|$ and, for every point $\mathfrak{p} \in |D|$, $F_D(z)$ is represented in a sufficiently small neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ as

$$F_D(z) = U(z)\Pi_k\{f_{k\mathfrak{p}}(z)\}^{m_k} \qquad (U(z) \neq 0),$$

where $U(z)$ is a non-vanishing regular analytic function defined in $N(\mathfrak{p})$. The absolute value

$$|F_D(z)| = |c| \exp 2\pi \{(-1)^{n(n+1)/2}/(n-1)!\}\ \Lambda^{n-1}\gamma[D](x)$$

of $F_D(z)$ is one-valued on $\mathfrak{M}$; thus $F_D(z)$ is multiplicative. If one prolongs $F_D(z)$ along an arbitrary closed curve ζ, then $F_D(z)$ is multiplied by the factor

$$\chi_D(\zeta) = \exp 2\pi i \{I(\zeta, C) + \textstyle\int_C w^*[\zeta]\} \qquad (\partial C = D), \tag{4.1}$$

where C is a $(2n-1)$-chain with $\partial C = D$ and $w[\zeta]$ means the harmonic 1-form of the first kind associated with the 1-cycle ζ in the sense of Hodge's Theorem.

As was already mentioned in §1, a multiplicative meromorphic function is determined by its divisor uniquely up to a multiplicative constant. Hence we get, as a corollary of the above theorem, the following

THEOREM 7. *A meromorphic function $F(z)$ with the divisor $D = \partial C$ is one-valued if and only if the congruence*

$$I(\zeta, C) + \int_C w^*[\zeta] \equiv 0 \pmod 1$$

holds for every 1-cycle ζ with integral coefficients.

This theorem can be considered as a generalization of Abel's Theorem [13, pp. 126-127] in the classical theory of Riemann surfaces.

Considered as a functional of 1-cycles ζ with integral coefficients, $\chi_D = \chi_D(\zeta)$ is a character of the 1-homology group $H^1(\mathfrak{M})$ of $\mathfrak{M}$ over the additive group of all integers. Then χ_D will be called the *integral character* [13, p. 125] associated with the divisor D.

As a simplest example, let us consider a "torus" $\mathfrak{T}$ obtained from the n-dimensional complex vector space $\{z;\ z = (z^1, z^2, \ldots, z^n)\}$ by identifying points which are congruent to each other with respect to the discrete subgroup generated by $2n$ linearly independent vectors $\pi_k = (\pi_k{}^\alpha)$ $(k = 1, 2, \ldots, 2n)$. The Kählerian metric

$$ds^2 = \Sigma_{\alpha=1}^{n}|dz^\alpha|^2$$

defined in the vector space is invariant under every translation

$$T_v\colon z^\alpha \to z^\alpha + v^\alpha;$$

hence it can be considered as a Kählerian metric attached to $\mathfrak{T}$. Putting

$$\pi_k^a = \pi_k^j + i\pi_k^{n+j} \qquad (j = a;\ a = 1, 2, \ldots n),$$

$$\pi_{ka} = \tfrac{1}{2}\bar{\pi}_k^a = \tfrac{1}{2}(\pi_k^j - i\,\pi_k^{n+j}) \qquad (j = a),$$

we introduce real components π_k^j and covariant components π_{ka} of π_k. Again, for simplicity's sake, we assume that $|\pi_k^j| = 1$, so that the volume of $\mathfrak{T}$ with respect to the metric ds^2 is equal to 1. The point $t\pi_k$ describes a closed curve in $\mathfrak{T}$ when the real parameter t moves from 0 to 1 which will be denoted by ζ_k. Then $\{\zeta_1, \zeta_2, \ldots, \zeta_{2n}\}$ constitutes a base of the 1-homology group $H^1(\mathfrak{T})$ of $\mathfrak{T}$. A base of the space $\mathfrak{E}^1$ of all harmonic 1-forms of the first kind is given simply by $\{dx^1, dx^2, \ldots\ dx^{2n}\}$; hence we have

$$w^j(x, \xi) = \Sigma_{k=1}^{2n} dx^k d\xi^k.$$

The harmonic 1-form $w[\zeta_k]$ associated with ζ_k is therefore represented as

$$w[\zeta_k](x) = \pi_{ka}dz^a + \bar{\pi}_{ka}d\bar{z}^a.$$

Now, suppose an irreducible closed analytic surface Γ in $\mathfrak{T}$ to be given and consider the $(2n-2)$-cycle

$$D = m\Gamma_v - m\Gamma, \quad \Gamma_v = T_v\Gamma,$$

m being a positive integer. Obviously D is homologous to zero; moreover, constructing the "cylinder"

$$C(v, \Gamma) = \{tv + z;\ 0 \leqq t \leqq 1,\ z \in \Gamma\}$$

over the surface Γ, we have

$$D = m\partial C(v, \Gamma).$$

From (4.1) follows therefore

$$\chi_D(\zeta) = \exp 2\pi i \left\{ m \int_{C(v,\,\Gamma)} w^*[\zeta] \right\}.$$

By a simple calculation, we obtain

$$\int_{C(v,\,\Gamma)} w^*[\zeta_k] = i \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(\bar{\pi}_{k\alpha}\bar{v}^\beta - \pi_{k\beta}v^\alpha), \tag{4.2}$$

where

$$A_{\alpha\beta} = \left(\frac{i}{2}\right)^{n-1} \int_\Gamma (-1)^{n+\alpha+\beta}[dz^1 \ldots dz^{\alpha-1}dz^{\alpha+1} \ldots dz^n d\bar{z}^1 \ldots d\bar{z}^{\beta-1}d\bar{z}^{\beta+1} \ldots d\bar{z}^n].$$

$C(\pi_j, \Gamma)$ is a $(2n-1)$-cycle which is homologous to the direct product $\zeta_j \times \Gamma$. Hence, putting

$$Q_{jk} = i \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(\pi_{k\beta}\pi_j{}^\alpha - \bar{\pi}_{k\alpha}\bar{\pi}_j{}^\beta), \tag{4.3}$$

we get by (4.2)

$$Q_{jk} = \int_{C(\pi_j,\,\Gamma)} w^*[\zeta_k] = I(\zeta_j \times \Gamma, \zeta_k) = I(\Gamma, \zeta_j \times \zeta_k);$$

thus Q_{jk} are integers. Finally, putting

$$v^\alpha = \Sigma_{j=1}^{2n} \delta^j \pi_j^\alpha,$$

δ^j being real numbers, we get from (4.2) and (4.3)

$$\int_{C(v,\,\Gamma)} w^*[\zeta_k] = \Sigma_j Q_{jk} \delta^j$$

and thus obtain the *formula*

$$\chi_D(\zeta_k) = \exp\, 2\pi i \left\{\Sigma_j Q_{jk} m \delta^j\right\}, \tag{4.4}$$

where $Q_{jk} = I(\Gamma, \zeta_j \times \zeta_k)$.

From this formula we can deduce several conclusions concerning meromorphic functions on $\mathfrak{T}$. First, since Q_{jk} are integers, we can choose, for given Γ, the vector v and the integer m so that $\Gamma_v \neq \Gamma$ and $\Sigma_j Q_{jk} m \delta^j \equiv 0 \pmod 1$. Hence *if $\mathfrak{T}$ contains an irreducible closed analytic surface Γ then there exists on $\mathfrak{T}$ a one-valued meromorphic function with the divisor of the type $D = m\Gamma_v - m\Gamma$.* As is well known, we can choose the periods π_k so that there exists on $\mathfrak{T}$ no one-valued meromorphic function other than constants. *Such $\mathfrak{T}$ contains therefore no closed analytic surface.* On the other hand, in case Γ is not a "cylindrical surface,"[17] we have

$$|Q_{jk}| = |A_{\alpha\beta}|^2 \neq 0,$$

and therefore we can choose v and m so that $\Gamma_v \neq \Gamma$ and that χ_D coincides with an arbitrarily preassigned character χ of $H^1(\mathfrak{T})$. Thus, *if $\mathfrak{T}$ contains an irreducible closed analytic surface Γ which is not cylindrical, then there exists on $\mathfrak{T}$ a multiplicative meromorphic function with the divisor of the type $D = m\Gamma_v - m\Gamma$ whose "multiplier" χ_D coincides with an arbitrarily preassigned χ of $H^1(\mathfrak{T})$.*

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